CHAPTER 7. WAVE MOTION

7-1 ONE-DIMENSIONAL WAVE

✓ A wave → Disturbance of medium
It travels with no change in its shape

Longitudinal wave : Displacement of medium ∥ Propagation direction
Transverse wave : Displacement of medium ⊥ Propagation direction

(a)

(b)

FIGURE 7-1

✓ A wave transports energy in space
But medium does not advance
→ Wave propagates at a great speed.

✓ Fig. 7-1(b)
Disturbance varies in time and space
\[ \psi = f(x,t) \] : Wavefunction (7-1)

↑
Wave shape, propagation direction, velocity, and so on.
Hold time or hold position

\[ \uparrow \]

Taking a picture \[ \uparrow \]
Profile Oscillation in time

\[ \psi \]

\[ f(x) \text{ at } t = 0 \rightarrow f(x) \text{ at } t = t_i \]

\[ \Delta x = v t_i \]

\[ \begin{align*}
  f(x) \text{ translated by } \Delta x = v t_i & \rightarrow \bar{f}(x) \\
  \bar{f}(x) = f(x - v t_i) 
\end{align*} \] (7-2)

A wave traveling in +x-direction

\[ \psi = f(x - v t) \] (7-3)

A wave traveling in -x-direction

\[ \psi = f(x + v t) \] (7-4)

✓ Wave motion is explained by differential wave equation

\[ \frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} : 1\text{-D} \] (7-5)

Velocity

The solution

\[ \psi = f(x \pm v t), \text{ or } \psi = f(vt \pm x) = g(t \pm x / v) \] (7-6)

Arbitrary
Example 7-1
Sketch wavefunctions, $\psi_1 = f(x-ut)$ and $\psi_2 = f(vt-x)$, at time $t = t_1$ for $f(x)$ given as in Fig. 7-3.

Solution
Both wavefunctions are solutions of the same differential wave equation. If $f$ were a cosine function, which is an even function, we could not distinguish between two waves, $\cos(x - ut)$ and $\cos(vt - x)$, by just looking at their profiles.

Example 7-2
Is $\psi$ a traveling wave or not?
(a) $\psi = (ax - bt)^2$
(b) $\psi = \sin(ax^2 - bt^2)$
where $a$ and $b$ are constants.

Solution
(a) Derivatives of $\psi$
$$\frac{\partial^2 \psi}{\partial x^2} = 2a^2$$ and $$\frac{\partial^2 \psi}{\partial t^2} = 2b^2$$
Inserting them in Eq. (7-5)
\[
\frac{\partial^2 \psi}{\partial t^2} \frac{\partial^2 \psi}{\partial x^2} = b^2 / a^2
\]
\[= \nu^2\]
Since \( \nu = b / a \) is a constant, \( \psi \) is a traveling wave.

(b) Derivatives of \( \psi \)
\[
\frac{\partial^2 \psi}{\partial x^2} = 2a \cos((ax^2 - bt^2)) - (2ax)^2 \sin((ax^2 - bt^2))
\]
\[
\frac{\partial^2 \psi}{\partial t^2} = -2b \cos((ax^2 - bt^2)) - (4bt)^2 \sin((ax^2 - bt^2))
\]

Inserting them in Eq. (7-5)
\[
\frac{\partial^2 \psi}{\partial t^2} \frac{\partial^2 \psi}{\partial x^2} \neq \text{constant}
\]
Since the ratio, which corresponds to \( \nu^2 \), is not a constant, \( \psi \) is not a traveling wave.

7-1.1 Harmonic Wave

It has sine or cosine profile
\[
\psi = A \cos(k(x - vt)) = A \cos(\omega t)
\]
(7-7)

\( A \), amplitude
\( k \), propagation constant
\( \omega \), angular frequency, \( \omega = 2\pi f \)

Phase velocity
\[
\nu_p = \frac{\omega}{k}
\]
(7-8)

Harmonic wave is periodic \( \rightarrow \) Spatial period or wavelength, \( \lambda \)
Temporal period \( \tau \)

A harmonic wave
\( \rightarrow \) Sinusoidal steady-state
Crests represent a phase \( 2\pi \), not \( 2\pi m \)

Periodicity requires
\[
A \cos(k(x \pm \lambda) - \omega t) = A \cos(kx - \omega t)
\]
(7-9a)
\[
A \cos(kx - \omega (t \pm \tau)) = A \cos(kx - \omega t)
\]
(7-9b)
\( \rightarrow \)
\[
k\lambda = 2\pi, \ \omega \tau = 2\pi
\]
(7-10)

\[
k = \frac{2\pi}{\lambda}
\]
(7-11a)
\[
\omega = \frac{2\pi}{\tau} = 2\pi f
\]
(7-11b)
\( \rightarrow \) \( \nu = f \lambda \)
(7-12)
Harmonic wave at $t = t_1$.
The inset → Wave as a function of $t$

Reversed and delayed

(No leading edge under sinusoidal steady-state condition)

Example 7-3
Given a harmonic wave $\psi = -4 \cos 2\pi(0.2x - 3t)$, find
(a) amplitude
(b) direction of propagation
(c) wavelength
(d) temporal period
(e) frequency
(f) phase velocity

Solution
(a) $4$
(b) $+x$-direction
(c) $k = \frac{2\pi}{\lambda} = 2\pi \times 0.2$
\[\lambda = 5[m]\]
(d) \( \omega = \frac{2\pi}{\tau} = 2\pi \times 3 \)
\( \tau = \frac{1}{3} \text{[sec]} \)

(e) \( \omega = 2\pi f = 2\pi \times 3 \)
\( f = 3 \text{[1/sec]} \)

(f) \( \nu_p = \frac{\omega}{k} = f\lambda = 15 \text{[m/s]} \)

**Example 7-4**

A wave generator at \( x = 0 \) produces a harmonic wave with phase velocity \( \nu_p \). Find the wavefunction in terms of cosine function.

**Solution**

Wavefunction in general form
\( \psi = A \cos (kx - \omega t + \delta) \)
where \( \delta \) is the initial phase

Consider a reference profile taken at \( t = t_1 \)
\( \psi_o = A \cos (kx - \omega t_1) \)
which has a crest at \( x = x_1 = (\omega / k)t_1 = \nu_p t_1 \).

\( \psi \) in Fig. 8-6 (a) is displaced to the left of \( \psi_o \) by \( \lambda / 4 \)
\( \psi = \psi_o (x + \lambda / 4, t_1) = A \cos \left[k(x + \lambda / 4) - \omega t_1\right] \)
\[ = A \cos \left[kx - \omega t_1 + \pi / 2\right] \]  
(7-13a)

\( \psi \) in Fig. 8-6 (b) is displaced to the right of \( \psi_o \) by \( \lambda / 4 \)
\( \psi = \psi_o (x - \lambda / 4, t_1) = A \cos \left[k(x - \lambda / 4) - \omega t_1\right] \)
\[ = A \cos \left[kx - \omega t_1 - \pi / 2\right] \]  
(7-13b)

The initial phase corresponds to the initial state of the wave being produced by the generator.
Time variation of $\psi$ at $x = x_i$.

![Wave Motion](image)

**FIGURE 7-7**

Let us compare $\psi$ in Fig. (7-7) with the reference wave $\psi_o = A \cos (kx_i - \omega t)$, which has a crest at $t = t_i = kx_i / \omega = x_i / v_p$.

$\psi$ in Fig. 7-7 (a) is displaced to the right of $\psi_o$ by $\tau / 4$

$$\psi = \psi_o (x_i, t - \tau / 4) = A \cos \left[ kx_i - \omega \left( t - \tau / 4 \right) \right]$$

$$= A \cos \left[ kx_i - \omega t + \pi / 2 \right]$$

(7-14a)

$\psi$ in Fig. 7-7 (b) is displaced to the left of $\psi_o$ by $\tau / 4$

$$\psi = \psi_o (x_i, t + \tau / 4) = A \cos \left[ kx_i - \omega \left( t + \tau / 4 \right) \right]$$

$$= A \cos \left[ kx_i - \omega t - \pi / 2 \right]$$

(7-14b)

Since $x_i$ and $t_i$ are arbitrary, and the harmonic wave extends from $-\infty$ to $\infty$, they are replaced by $x$ and $t$. Then, Eq. (7-14) is the same as Eq. (7-13).

### 7-1.2 Complex Form of Harmonic Wave

Complex exponential $\rightarrow$ Easy to combine harmonic waves

Easy to handle phase and impedance

![Euler formula](image)

(7-15)

Cosine and sine as

$$\cos \theta = \text{Re} \left[ e^{i\theta} \right] = \frac{1}{2} \left( e^{i\theta} + e^{-i\theta} \right)$$

: real part

(7-16a)

$$\sin \theta = \text{Im} \left[ e^{i\theta} \right] = \frac{1}{2i} \left( e^{i\theta} - e^{-i\theta} \right)$$

: imaginary part

(7-16b)
A harmonic wave in **real instantaneous form**

\[ \psi = A \cos(kx - \omega t + \delta) = \text{Re}\left[ Ae^{ikx-\omega t+i\delta} \right] \]  

(7-17)

A harmonic wave in **complex form**

\[ \psi(x, t) = Ae^{ikx-\omega t+i\delta} \]  

(7-18)

Rewriting Eq. (7-18)

\[ \psi(x, t) = (Ae^{i\delta})e^{ikx}e^{-i\omega t} = [\hat{A}e^{i\delta}]e^{-i\omega t} \]

(7-19)

↑

**Scalar complex amplitude**, \( \hat{A} \),

A scalar and a complex number

No change of \( \omega \) in linear media

A harmonic wave in time-independent complex form

\[ \psi = \hat{A}e^{ikx} \]  

(7-20)

The real instantaneous wavefunction is obtained by multiplying \( \psi \) with \( e^{-i\omega t} \) and then taking the real part.

Use of \(-i\) instead of \(i\)

\(-i\) is used more in engineering

To avoid any confusions, we express \(-i\) as \(-j\), where \( j = \sqrt{-1} \)

A harmonic wave in complex form

\[ \psi(x, t) = \hat{A}e^{i\omega t-kx} \]  

(7-21)

Ignoring \( e^{i\omega t} \), a harmonic wave in **phasor form**

\[ \psi = \hat{A}e^{-j\omega t} \]  

(7-22)

The real instantaneous wavefunction is obtained by multiplying the phasor with \( e^{i\omega t} \) and taking the real part.

Comparison of two expressions

\[ \psi = \hat{A}e^{ikx} \quad \psi = \hat{A}e^{-j\omega x} \]

↑

Spatial distribution of phase. \quad Spatial motion of wave.

Time delays of wave. \quad Impedance

(Phase increases with increasing \( x \)) \quad (Positive phase means leading in time phase)

An expression should be used consistently in a given problems
Example 7-5
A harmonic wave \( \psi = A_0 e^{ikx} \) travels in free space. After it passes through a lossless medium of \( k' = nk \) in \( 0 \leq x \leq d \), it is delayed by \( \lambda / 4 \), where \( \lambda \) is the wavelength in free space. Determine wavefunction
(a) \( \psi_1 \) in the medium
(b) \( \psi_2 \) after the medium.
(c) Find \( n \) in terms of \( d \) and \( \lambda \).

\[
\psi = A_0 e^{ikx}
\]

\[
k' = nk
\]

\[
\psi = \psi_1 = A_0 e^{ikx}
\]

\[
\psi_2 \text{ lags behind } \psi \text{ by } \lambda / 4 , \text{ or } \psi_2 \text{ is displaced to the left by } \lambda / 4
\]

\[
\psi_2 = A_0 e^{ik(x+d/4)}
\]

\[
\psi_1 (x = d) = \psi_2 (x = d)
\]

or

\[
A_0 e^{i(kd)} = A_0 e^{i(kd+\lambda/4)}
\]  \( (7-23) \)

We obtain from Eq. (7-23)

\[
d = \frac{d + \lambda}{4}
\]
Example 7-6
A harmonic wave $\psi = A_0 e^{-j k x}$ impinges on a material at $x = x_3$, and undergoes a reflection. The reflected wave $\psi'$ propagates in $-x$-direction. Boundary condition requires that $\psi'(x_3) / \psi(x_3) = 0.5$ at $x = x_3$.
(a) Find the phasor of $\psi'$
(b) Find the total waves at $x = x_1$ and $x = x_2$
(c) What time phase does the total wave at $x = x_1$ lags behind that at $x = x_2$?

![Illustration of wave reflection](image)

FIGURE 7-9

(a) Incident wave at $x = x_3$
$\psi(x_3) = A_0 e^{-j k x_3}$  

(7-24)

Reflected wave in general form
$\psi'(x) = A'_0 e^{j k x + \delta}$

(7-25)

Applying boundary condition at $x = x_3$,
$\psi'(x_3) = \frac{1}{2} \psi(x_3)$

or
$A'_0 e^{j k x_3 + \delta} = \frac{1}{2} A_0 e^{-j k x_3}$

(7-26)

Inserting $x_3 = 2.25 \lambda$ and $k = 2\pi / \lambda$ in Eq. (7-26)
$A'_0 e^{j 4.5\pi + \delta} = \frac{1}{2} A_0 e^{-j 4.5\pi}$

Ignoring multiples of $2\pi$ in the phase
$A'_0 = \frac{1}{2} A_0$ and $\delta = -\pi$

(7-27)

Wavefunction of the reflected wave
$\psi' = \frac{1}{2} A_0 e^{j k x - j \delta}$

(7-28)
(b) Total wave at \( x = x_1 = 0.3\lambda \)
\[
\psi_\tau(x_1) = \psi(x_1) + \psi'(x_1) = A_x e^{-j k x_1} + \frac{1}{2} A_y e^{j k x_1} = A_x e^{-j 0.6\pi} + \frac{1}{2} A_y e^{j 0.4\pi}
\]
or
\[
\psi_\tau(x_1) = 1.43 A_x e^{-j 1.68}\pi
\]  
(7-29)

Total wave at \( x = x_2 = 1.25\lambda \)
\[
\psi_\tau(x_2) = \psi(x_2) + \psi'(x_2) = A_x e^{-j k x_2} + \frac{1}{2} A_y e^{j k x_2} = A_x e^{-j 2.5\pi} + \frac{1}{2} A_y e^{j 1.5\pi}
\]
or
\[
\psi_\tau(x_2) = 1.5 A_x e^{-j 0.5\pi}
\]  
(7-30)

(c) Phase difference between Eqs. (7-29) and (7-30)
\[
\Delta \theta = \theta_1 - \theta_2 = -1.68 - (-0.5\pi) = -0.11 \ [\text{rad}]
\]

The total wave at \( x = x_1 \) lags behind the total wave at \( x = x_2 \) by 0.11 radian in time phase.
(A negative phase simply means a lag in time phase)

We have used \( e^{j k x} \) for a harmonic wave in Example 7-5 and \( e^{-j k x} \) in Example 7-6. However, any expression can solve both problems equally well. As may be seen from the examples, the use of \( e^{j k x} \) makes it easier to visualize propagation of the wave in space, and the use of \( e^{-j k x} \) is advantageous in specifying time delays of the wave at points in space.

7-2 THREE-DIMENSIONAL PLANE WAVE

**Three-dimensional differential wave equation**
\[
\nabla^2 \psi = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}
\]  
(7-31)

Laplacian operator in Cartesian coordinates
\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}
\]  
(7-32)

Solution of the wave equation
\[
\psi = A_x \cos \left( (k_x x + k_y y + k_z z) \pm \omega t + \varphi \right) : \varphi \text{ is constant}
\]  
(7-33)

In complex form
\[
\psi = A e^{j \mathbf{r} \cdot \mathbf{m} + j \omega t} \]  
(7-34)

\[
\hat{A} = A_x \exp(j \varphi), \quad : \text{Scalar complex amplitude}
\]
\[
\mathbf{k} = k_x \mathbf{a}_x + k_y \mathbf{a}_y + k_z \mathbf{a}_z \quad : \text{Wavevector}
\]  
(7-35a)
\[
|\mathbf{k}| = \sqrt{k_x^2 + k_y^2 + k_z^2} = \frac{2\pi}{\lambda} \quad : \lambda \text{ is the wavelength}
\]  
(7-35b)
\[
\mathbf{r} = x \mathbf{a}_x + y \mathbf{a}_y + z \mathbf{a}_z \quad : \text{Position vector}
\]  
(7-36)
\textbf{✓} \textbf{k} \cdot \textbf{r} \text{ determines spatial distribution of the phase}

Points of an equal phase $\rightarrow$ \textbf{Phase front} or \textbf{wavefront}

A wave with plane wavefront $\rightarrow$ Plane wave.

\[
\textbf{k} \cdot \textbf{r} = \alpha \quad \rightarrow \quad \text{Phase at } \textbf{r} \text{ is } \alpha
\]

\[k \cdot r = \alpha \quad \rightarrow \quad r_0 \text{ is one of } \textbf{r} \text{’s satisfying Eq. (7-37)} \quad (7-38)\]

Combining Eqs. (7-37) and (7-38)

\[
\textbf{k} \cdot (\textbf{r} - r_0) = 0
\]

\[A \text{ plane perpendicular to } \textbf{k} \text{ and including } r_0 \]

\[\text{FIGURE 7-10}\]

\textbf{✓} Wavefronts are periodic with a period $\lambda$

Change position by $\lambda$ in $\textbf{a}_k$ direction

\[
\hat{A}e^{\textbf{k} \cdot (\textbf{r} + \lambda \textbf{a}_k)} = \hat{A}e^{\textbf{k} \cdot \textbf{r}}e^{i\lambda} = \hat{A}e^{\textbf{k} \cdot \textbf{r}}
\]

\[k \lambda = 2\pi \quad (7-40)\]

\[k = \frac{2\pi}{\lambda} \quad : \text{wavenumber} \quad (7-41)\]
A plane wave $\rightarrow$ Parallel wavefronts moving in $\mathbf{a}_k$-direction  
(Not easy to draw on a paper) 
A simple cosine 

Each point $\rightarrow$ A plane wavefront  
Ordinate : magnitude  
Abscissa : phase

$k$ defines the propagation direction and period of the wavefront.

FIGURE 7-12
Example 7-7

An incident plane wave $\mathbf{k}_i$ is partly transmitted across the interface at $xy$-plane, $\mathbf{k}_t$, while the rest is reflected, $\mathbf{k}_r$. The boundary condition requires that three waves should have the same spatial period along $y$-axis, and that the angles of reflection and transmission, $\theta_r$ and $\theta_i$, should be equal. When $|\mathbf{k}| = 2\pi / \lambda$, determine the wavelength of the transmitted wave.

Solution

Spatial period of the incident wave along $y$-axis
$$\lambda / \sin \theta_i \quad (7\text{-}42)$$

Let $\lambda_i$ be the wavelength of the transmitted wave.

Spatial period of the transmitted wave along $y$-axis
$$\lambda_i / \sin \theta_i \quad (7\text{-}43)$$

Equating Eqs. (7-42) and (7-43)
$$\lambda_i = \lambda \frac{\sin \theta_i}{\sin \theta_i}$$

The wavelength of the transmitted wave is smaller than that of the incident wave, if $\theta_i < \theta_r$. 
7-3 ElectroMagnetic Plane Wave

Maxwell’s equations in free space, \( \rho = J = 0 \)

\[
\begin{align*}
\nabla \times \mathbf{E} &= -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \\
\nabla \times \mathbf{H} &= \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \\
\nabla \cdot \mathbf{E} &= 0 \\
\nabla \cdot \mathbf{H} &= 0
\end{align*}
\]  

(7-42a) (7-42b) (7-42c) (7-42d)

Where

\( \mathbf{E} \) and \( \mathbf{H} \) are instantaneous electric and magnetic field intensities

\( \varepsilon_0 \) and \( \mu_0 \) are permittivity and permeability of free space

In Cartesian coordinates

\[
\begin{align*}
\mathbf{E}(r, t) &= E_x(r, t) \mathbf{a}_x + E_y(r, t) \mathbf{a}_y + E_z(r, t) \mathbf{a}_z \\
\mathbf{H}(r, t) &= H_x(r, t) \mathbf{a}_x + H_y(r, t) \mathbf{a}_y + H_z(r, t) \mathbf{a}_z
\end{align*}
\]

(7-43a) (7-43b)

\( \mathbf{r} \) is the position vector

\( E_x, a_x \) are a scalar component of \( \mathbf{E} \)

\( E_x, a_x \) are a vector component of \( \mathbf{E} \)

Taking the curl of Eq. (7-42a) and inserting Eq. (7-42b)

\[
\nabla \times \nabla \times \mathbf{E} = -\varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}
\]

(7-44)

Using a vector identity, \( \nabla \times \nabla \times \mathbf{A} = \nabla \nabla \mathbf{A} - \nabla^2 \mathbf{A} \), and inserting Eq. (7-42c)

\[
\nabla^2 \mathbf{E} = \varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}
\]

: Three-dimensional vector differential wave equation

(7-45)

Three vector components of \( \mathbf{E} \) are mutually exclusive

\[
\begin{align*}
\nabla^2 E_x &= \varepsilon_0 \mu_0 \frac{\partial^2 E_x}{\partial t^2} \\
\nabla^2 E_y &= \varepsilon_0 \mu_0 \frac{\partial^2 E_y}{\partial t^2} \\
\nabla^2 E_z &= \varepsilon_0 \mu_0 \frac{\partial^2 E_z}{\partial t^2}
\end{align*}
\]

(7-46a) (7-46b) (7-46c)

Solutions are given by plane waves

\[
\begin{align*}
E_x(r, t) &= E_1 \cos(k \cdot r - \omega t + \phi_x) \\
E_y(r, t) &= E_2 \cos(k \cdot r - \omega t + \phi_y) \\
E_z(r, t) &= E_3 \cos(k \cdot r - \omega t + \phi_z)
\end{align*}
\]

A single solution

(7-47a) (7-47b) (7-47c)

The same \( k \) and \( \omega \)

Solution of the three-dimensional vector differential wave equation

\[
\mathbf{E}(r, t) = \text{Re}[E_0 e^{i(k \cdot r - \omega t)}]
\]

(7-48)
\( E_0 \) is the **vector complex amplitude**

\[
\uparrow
\]

A vector whose scalar components are complex quantities.

\[
E_0 = E_{0x} e^{i\phi_x} \mathbf{a}_x + E_{0y} e^{i\phi_y} \mathbf{a}_y + E_{0z} e^{i\phi_z} \mathbf{a}_z
\]  \hspace{1cm} (7-49a)

\( \phi_x = \phi_y = \phi_z = \phi \), in many cases

\[
E_0 = E_0 e^{i\phi} \mathbf{a}_E
\]  \hspace{1cm} (7-49b)

\( E_0 \), amplitude

\( \phi \), phase angle

\( \mathbf{a}_E \), unit vector

Phase velocity

\[
u_p = \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \approx 3 \times 10^8 \text{ [m / s]} \]  \hspace{1cm} (7-50)

which can be checked by by substituting Eq. (7-48) in Eq. (7-45)

\( \checkmark \)  Differential wave equation for magnetic field intensity

\[
\nabla^2 \mathbf{\mathcal{H}} = \varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{\mathcal{H}}}{\partial t^2}
\]  \hspace{1cm} (7-51)

plane wave solution

\[
\mathbf{\mathcal{H}}(r, t) = \text{Re} \left[ \mathbf{H} e^{i \mathbf{a} \cdot r - \omega t} \right]
\]  \hspace{1cm} (7-52)

\( \mathbf{E} \) and \( \mathbf{H} \) are interrelated \( \rightarrow \) Solve either Eq.(7-45) or Eq. (7-51)

The other is obtained from Maxwell’s equations.

**7-3.1 TRANSVERSE WAVE**

\( \mathbf{E} \) generates \( \mathbf{H} \)

\( \uparrow \)

\( \mathbf{H} \) generates \( \mathbf{E} \)

\( \uparrow \)

Perpendicular to \( \mathbf{E} \)

Perpendicular to \( \mathbf{H} \)

\( \mathbf{E} \) and \( \mathbf{H} \) are interrelated, symmetric and perpendicular to each other

\( \rightarrow \) Propagation direction should be perpendicular to both \( \mathbf{E} \) and \( \mathbf{H} \)

A plane wave propagating along \( z \)-axis.

\( \rightarrow \) \( \mathbf{E} \) has no change along \( x \)- and \( y \)-axes

\[
\frac{\partial \mathbf{E}(r, t)}{\partial x} = \frac{\partial \mathbf{E}(r, t)}{\partial y} = 0
\]  \hspace{1cm} (7-53)

or

\[
\mathbf{E}(r, t) = E_x(z, t) \mathbf{a}_x + E_y(z, t) \mathbf{a}_y + E_z(z, t) \mathbf{a}_z
\]  \hspace{1cm} (7-54)
Inserting Eq. (7-54) in Eq. (7-42c)
\[ \mathbf{v} \cdot \mathbf{\varepsilon} = \frac{\partial \mathcal{E}_x}{\partial x} + \frac{\partial \mathcal{E}_y}{\partial y} + \frac{\partial \mathcal{E}_z}{\partial z} \]
\[ = \frac{\partial \mathcal{E}_x}{\partial z} = 0 \]  
\[ \mathcal{E}_x \text{ is constant, not a traveling wave} \]
Thus, \( \mathcal{E}_x = 0 \) (7-56)

Then, we have
\[ \mathbf{\varepsilon} = \mathcal{E}_x(z,t) \mathbf{a}_x + \mathcal{E}_y(z,t) \mathbf{a}_y \]  
Perpendicular to the propagation direction of \( \mathbf{\varepsilon} \)

Inserting Eq. (7-57) in Eq. (7-42a)
\[ -\frac{\partial \mathcal{E}_x}{\partial z} \mathbf{a}_x + \frac{\partial \mathcal{E}_y}{\partial z} \mathbf{a}_y = -\mu_0 \frac{\partial \mathcal{H}_x}{\partial t} \mathbf{a}_x + \mu_0 \frac{\partial \mathcal{H}_y}{\partial t} \mathbf{a}_y - \mu_0 \frac{\partial \mathcal{H}_z}{\partial t} \mathbf{a}_z \]
\[ \rightarrow \mathcal{H}_x = 0 \]  
(7-59)

From Eqs. (7-56) and (7-59)
\[ \rightarrow \text{The electromagnetic wave is a transverse wave.} \]

**Example 7-8**

Find magnetic field intensity of a plane wave that propagates in free space with an electric field intensity \( \mathbf{\varepsilon} = E_z \mathbf{a}_x \cos (kz - \omega t) \).

**Solution**

Inserting \( \mathbf{\varepsilon} \) in Eq. (7-42a)
\[ \mathbf{a}_y E_z \frac{\partial}{\partial z} \cos (kz - \omega t) = -\mu_0 \frac{\partial}{\partial t} \left( \mathcal{H}_x \mathbf{a}_x + \mathcal{H}_y \mathbf{a}_y + \mathcal{H}_z \mathbf{a}_z \right) \]

It reduces to
\[ kE_z \sin (kz - \omega t) = \mu_0 \frac{\partial \mathcal{H}_y}{\partial t} \]

Integrating both sides with respect to \( t \)
\[ \mathcal{H}_y = \sqrt{\frac{\epsilon_0}{\mu_0}} E_z \cos (kz - \omega t) \]

Thus,
\[ \mathcal{H} = \sqrt{\frac{\epsilon_0}{\mu_0}} E_z \mathbf{a}_y \cos (kz - \omega t) \]

\( \mathcal{H} \) has the same form as \( \mathbf{\varepsilon} \), except for \( \sqrt{\frac{\epsilon_0}{\mu_0}} \) and the direction of the field. Note that \( \mathbf{\varepsilon} \) and \( \mathcal{H} \) are in phase. Also note that \( \mathbf{\varepsilon} \) and \( \mathcal{H} \) are perpendicular to each other, and that \( \mathbf{\varepsilon} \times \mathcal{H} \) points to the propagation direction of the wave.
FIGURE 7-13