CHAPTER 8. ELECTROMAGNETIC HARMONIC WAVE

Maxwell’s equations  →  Differential wave equation  →  Traveling wave
A harmonic wave

An arbitrary wave  →  A linear combination of harmonic waves.

Superposition principle.
Linear media, (ε, μ are constant)

Harmonic waves  →  A constant amplitude
Time-invariant

Sinusoidal steady-state

✓ Time-varying \( \boldsymbol{\mathcal{E}} \) and \( \boldsymbol{\mathcal{H}} \)  →  Harmonic waves in 3D-space.

\[ \cos(\mathbf{k} \cdot \mathbf{r} \pm \omega t), \cos(\omega t \pm \mathbf{k} \cdot \mathbf{r}) \]

Maxwell’s eqs.

\[ + \text{ phase change, phase lead} \]
\[ - \text{ phase change, phase lag} \]

✓ Subjects of this chapter
- Propagation harmonic waves
- Harmonic waves on an interface (Boundary condition)
- Introduction of \textit{phasor}

Time-invariant part of the complex form
of a harmonic wave.

8-1 THE PHASOR

In a medium of \( \rho_0 = 0 \), \( \mathbf{J} = 0 \) and constant \( \varepsilon \) and \( \mu \)

Maxwell’s equations

\[ \nabla \times \boldsymbol{\mathcal{E}} = -\mu \frac{\partial \boldsymbol{\mathcal{H}}}{\partial t} \quad (8-1a) \]
\[ \nabla \times \boldsymbol{\mathcal{H}} = \varepsilon \frac{\partial \boldsymbol{\mathcal{E}}}{\partial t} \quad (8-1b) \]
\[ \nabla \cdot \boldsymbol{\mathcal{E}} = 0 \quad (8-1c) \]
\[ \nabla \cdot \boldsymbol{\mathcal{H}} = 0 \quad (8-1d) \]

✓ Combining two curl equations

\[ \nabla^2 \boldsymbol{\mathcal{E}} = \mu_0 \frac{\partial^2 \boldsymbol{\mathcal{E}}}{\partial t^2} \quad (8-2) \]

Its solution

\[ \boldsymbol{\mathcal{E}}(\mathbf{r}, t) = \text{Re} \left[ \mathbf{E}_0 e^{-\mathbf{k} \cdot \mathbf{r} - j \omega t} \right] \]

Uniform plane wave

Harmonic wave

8-1

EM Harmonic Wave
Proprietary of Prof. Lee, Yeon Ho
where
\[ \mathbf{E}_o, \text{ vector complex amplitude} \]
\[ k = \omega \sqrt{\mu \varepsilon} a_k, \text{ wavevector} \]

Eq. (8-2) \[ \rightarrow |k|, \text{ but no } a_k \]
\[ \uparrow \]
From source or boundary condition

✓ In a medium of finite extent
The same differential wave eq. \[ \rightarrow \text{ Uniform plane waves} \]

Boundary condition \[ \rightarrow \text{ Linear combination} \]
(Different \( k \)'s)

A wave on a boundary
\[ \rightarrow \text{ A partial reflection} \]
\[ \rightarrow \text{ Total wave} = \text{ incident wave} + \text{ reflected wave} \]
\[ \mathcal{E}(r, t) = \text{Re} \left[ \mathbf{E}_i e^{-jk \cdot r} + \mathbf{E}_r e^{-jk \cdot r} e^{j\omega t} \right] \] (8-5)
\[ \uparrow \]
Solution of wave equation if \( |\mathbf{k}_i| = |\mathbf{k}_r| = \omega \sqrt{\mu \varepsilon}. \)

In general form
\[ \mathcal{E}(r, t) = \text{Re} \left[ \mathbf{E}(x, y, z) e^{j\omega t} \right] \] (8-6)
\[ \uparrow \]
**Phasor**, time invariant.
It assumes sinusoidal steady-state condition.
It shows relative phases, or time delays, in space.
It is useful in combining multiple harmonic waves.
It is valid in linear media.

✓ Differential wave eq. for \( \mathcal{H} \) has the same form as \( \mathcal{E} \)
\[ \mathcal{H}(r, t) = \text{Re} \left[ \mathbf{H}(x, y, z) e^{j\omega t} \right] \] (8-7)
\[ \uparrow \]
Phasor of \( \mathcal{H} \)

**Example 8-1**
Resolve a harmonic wave \( \mathbf{E} = a_y E_o \cos(k_x x) e^{-jk_z z} \) into two uniform plane waves and sketch their wavefronts.

**Solution**
Using Euler formula, we rewrite
\[
\mathbf{E} = a_y E_o \frac{1}{2} \left[ e^{jk_x x} + e^{-jk_x x} \right] e^{-jk_z z} \\
\mathbf{E} = a_y E_o \frac{1}{2} e^{-j(k_x x + k_z z)} + a_y E_o e^{-j(k_x x + k_z z)} \\
= \mathbf{E}' + \mathbf{E}''
\]
The right side is the sum of two uniform plane waves of wavevectors \( \mathbf{k}' = -k_z \mathbf{a}_z + k_x \mathbf{a}_x \) and \( \mathbf{k}'' = k_z \mathbf{a}_z + k_x \mathbf{a}_x \), respectively. Both waves have the same wave number 
\[ |\mathbf{k}| = |\mathbf{k}''| = \sqrt{k_x^2 + k_z^2} \]

![Diagram](image)

**FIGURE 8-1**

The black (or blue) parallel lines represent plane wavefronts, viewed from the top, of the wave of \( \mathbf{k}' \) (or \( \mathbf{k}'' \)). They correspond to phases \( 2\pi n \), where \( n \) are integers. The dots on \( x \) axis are the points at which wavefronts of the same phase meet, and therefore the total electric field intensity has the maximum amplitude \( E_0 \). The amplitude varies sinusoidally along \( x \) axis as is indicated by the term \( \cos (k_x x) \).

### 8-2 MAXWELL’S EQUATIONS IN PHASOR FORM

Inserting Eqs. (8-6) and (8-7) in Eq. (8-1a)
\[
\nabla \times \text{Re} \left[ \mathbf{E}(x, y, z) e^{j\omega t} \right] = -j \omega \mu \text{Re} \left[ \mathbf{H}(x, y, z) e^{j\omega t} \right]
\]
\[\rightarrow \text{Re} \left[ e^{j\omega t} \nabla \times \mathbf{E}(x, y, z) \right] = -j \omega \mu \text{Re} \left[ e^{j\omega t} j \omega \mathbf{E}(x, y, z) \right]\]

Curl, time derivative and real part are mutually exclusive

Ignore Re and \( \exp (j\omega t) \)

Faraday’s law in phasor form
\[\nabla \times \mathbf{E} = -j \omega \mu \mathbf{H}\]

Maxwell’s equations in phasor form with constant \( \varepsilon \) and \( \mu \)
\[
\nabla \times \mathbf{E} = -j \omega \mu \mathbf{H} \tag{8-9a}
\n\nabla \times \mathbf{H} = j \omega \varepsilon \mathbf{E} \tag{8-9b}
\n\n\nabla \cdot \mathbf{E} = 0 \tag{8-9c}
\n\n\nabla \cdot \mathbf{H} = 0 \tag{8-9d}
\]

\( \mathbf{E} \) and \( \mathbf{H} \) are time-invariant complex vectors
Combine two curl equations
\[ \nabla \times \nabla \times \mathbf{E} = -j \omega \mu \nabla \times \mathbf{H} = \omega^2 \mu \varepsilon \mathbf{E} \]  
(8-10)

Using a vector identity, \( \nabla \times \nabla \times \mathbf{E} = \nabla \nabla \cdot \mathbf{E} - \nabla^2 \mathbf{E} \), and Eq. (8-9c),
\[ \nabla^2 \mathbf{E} + k^2 \mathbf{E} = 0 \quad : \text{Vector Helmholtz's equation} \]  
(8-11)

\[ k = \omega \sqrt{\mu \varepsilon} = \frac{\omega}{v} = \frac{2\pi}{\lambda} \quad : \text{Wavenumber} \]  
(8-12)

where
- \( \omega \), angular frequency
- \( \mu \), permeability
- \( \varepsilon \), permittivity
- \( v \), velocity
- \( \lambda \), wavelength.

8-3 PLANE WAVE IN LOSSLESS DIELECTRIC

In Cartesian coordinates, vector Helmholtz’s equation
\[ \nabla^2 \left( E_x \mathbf{a}_x + E_y \mathbf{a}_y + E_z \mathbf{a}_z \right) + k^2 \left( E_x \mathbf{a}_x + E_y \mathbf{a}_y + E_z \mathbf{a}_z \right) = 0 \]  
(8-13)

Mutually exclusive

\[ \nabla^2 E_x + k^2 E_x = 0 \]  
(8-14a)
\[ \nabla^2 E_y + k^2 E_y = 0 \]  
(8-14b)
\[ \nabla^2 E_z + k^2 E_z = 0 \]  
(8-14c)

Rewriting Eq. (8-14a)
\[ \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0 \]  
(8-15)

Its solution
\[ E_x = \tilde{E}_{\omega \mathbf{a}} e^{\pm j(k_x x + k_y y + k_z z)} = \tilde{E}_{\omega \mathbf{a}} e^{\pm \mathbf{k} \cdot \mathbf{r}} \]  
(8-16)

\[ \mathbf{k} = k \mathbf{a}_x = a_x \sqrt{k_x^2 + k_y^2 + k_z^2} \quad \text{wavevector} \]  
(8-17)

Scalar complex amplitude

Similarly
\[ E_y = \tilde{E}_{\omega \mathbf{a}} e^{-\mathbf{k} \cdot \mathbf{r}} \]  
(8-18a)
\[ E_z = \tilde{E}_{\omega \mathbf{a}} e^{-\mathbf{k} \cdot \mathbf{r}} \]  
(8-18b)

Helmholtz’s equations in Eq. (8-14)
→ The same \( |\mathbf{k}| \) in Eqs. (8-16), (8-18a) and (8-18b)
But no restriction on \( \mathbf{a}_x \)
The same $a_k$ → Three solutions into a single wave

$$E = E_w a_x + E_y a_y + E_z a_z = (\tilde{E}_{sx} a_x + \tilde{E}_{sy} a_y + \tilde{E}_{sz} a_z) e^{-jkr}$$

or

$$E = E_w e^{-jkr}$$

(8-19)

where

$$E_w = \tilde{E}_{sx} a_x + \tilde{E}_{sy} a_y + \tilde{E}_{sz} a_z$$

$$= \tilde{E}_w a_x = E_o e^{j\phi} a_x$$

↑ Scalar complex amplitude

Vector complex amplitude

$E_o$, amplitude
$\phi$, phase angle

$E = E_w e^{-jkr}$, a uniform plane wave in 3D space.

$k$, wavevector
propagation direction and wavelength

$E_w$, vector complex amplitude
direction and amplitude of $E$

$r$, position vector

$-k \cdot r$, time-phase of the field

$E_w$ → $\tilde{E}_w a_x$

$\tilde{E}_{sx}$, $\tilde{E}_{sy}$ and $\tilde{E}_{sz}$ have the same $\phi$ phase angle

A single wave in an infinite medium

$\Rightarrow \phi = 0$

Transverse wave

Inserting Eq. (8-19) in Eq. (8-9c)

$$\nabla \cdot E = \nabla \cdot (E_w e^{-jkr}) = -j (k \cdot E_w) e^{-jkr} = 0$$

or

$$k \cdot E_w = 0$$

(8-21)

Propagation direction ⊥ Oscillation direction

$E_w$ is in the wavefront
\( \textbf{H} \) from Eq. (8-9a)
\[
\nabla \times \mathbf{E} = \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\hat{E}_x e^{-jkr} & \hat{E}_y e^{-jkr} & \hat{E}_z e^{-jkr} \\
\end{vmatrix} = \begin{vmatrix}
\mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\
-jk_x & -jk_y & -jk_z \\
\end{vmatrix} = -j\omega \mu \textbf{k} e^{-jkr} \tag{8-22} \]

or
\[
\mathbf{H} = \frac{1}{\eta}(\mathbf{a}_k \times \mathbf{E}) \tag{8-23} \]

\[
\eta = \frac{\mu}{\varepsilon} : \text{Intrinsic impedance} \tag{8-24} \]

Rewriting (8-23)
\[
\mathbf{H} = \frac{1}{\eta}(\mathbf{a}_k \times \mathbf{E}_o) e^{-jkr} = \mathbf{H}_o e^{-jkr} \tag{8-25} \]

\[\uparrow\]

Same form as \( \mathbf{E} \)

Inserting \( \mathbf{E}_o = \mathbf{E}_o \mathbf{a}_x \)
\[
\mathbf{H}_o = \frac{\mathbf{E}_o}{\eta}(\mathbf{a}_k \times \mathbf{a}_x) = \mathbf{H}_o \mathbf{a}_h \tag{8-26} \]

\[
\eta = \mathbf{E}_o / \mathbf{H}_o, \text{ in ohms [}\Omega]\]
\[
\eta_o = \sqrt{\mu_o / \varepsilon_o} = 120\pi \approx 377 [\Omega] \]

Lossless dielectrics \( \rightarrow \) \( \eta \) is real \( \rightarrow \) \( \mathbf{E} \) and \( \mathbf{H} \) are in phase

Lossy dielectrics \( \rightarrow \) \( \eta \) is complex \( \rightarrow \) \( \mathbf{E} \) and \( \mathbf{H} \) are out of phase

\( \textbf{H}_o \perp (\mathbf{E}_o \text{ and } \mathbf{k}) \), from Eq. (8-25)
\( \mathbf{E}_o \perp \mathbf{k} \), from Eq. (8-21)
\( \rightarrow \) \( \mathbf{E}_o \), \( \mathbf{H}_o \) and \( \mathbf{k} \) are mutually orthogonal

\[\text{Transverse wave!}\]

Real instantaneous expression
\[
\mathbf{E}(\mathbf{r}, t) = a_x E_o \cos (\omega t - \mathbf{k} \cdot \mathbf{r} + \phi) \tag{8-27a} \]
\[
\mathbf{H}(\mathbf{r}, t) = a_h \frac{E_o}{\eta} \cos (\omega t - \mathbf{k} \cdot \mathbf{r} + \phi) \tag{8-27b} \]

Unit vectors
\[
\mathbf{a}_k = \mathbf{a}_x \times \mathbf{a}_h \tag{8-28} \]
Example 8-2
A plane wave, \( \mathbf{E} = a_x E_o (1 + j) e^{-j\beta z} \), has a frequency of 2 GHz and propagates in a dielectric of \( \varepsilon_r = 4 \) and \( \mu_r = 1 \). Find
(a) \( \mathbf{E} \) and \( \mathbf{H} \)
(b) wavelength

Solution
(a) Rewriting the phasor
\[
\mathbf{E} = a_x E_o \sqrt{2} e^{j\omega t} e^{-j\beta z} \tag{8-29}
\]
The real instantaneous expression
\[
\mathbf{E} = \text{Re}\left[ a_x E_o \sqrt{2} e^{j\omega t} e^{-j\beta z} \right] = a_x E_o \sqrt{2} \cos\left( \omega t - k z + \frac{\pi}{4} \right)
\]
Inserting Eq. (8-29) in Eq. (8-23)
\[
\mathbf{H} = \frac{1}{\eta} \left( a_y \times \mathbf{E} \right) = \frac{1}{\eta} \left( a_y \times a_x \right) E_o \sqrt{2} e^{j\omega t} e^{-j\beta z}
\]
where \( a_y = a_x \).

Intrinsic impedance
\[
\eta = \sqrt{\frac{\mu_r}{\varepsilon_r}} = 188.5 \, \Omega.
\]

Real instantaneous expression
\[
\mathbf{H} = \text{Re}\left[ \frac{1}{\eta} a_y E_o \sqrt{2} e^{j\omega t} e^{-j\beta z} \right] = a_y \frac{E_o}{\eta} \sqrt{2} \cos\left( \omega t - k z + \frac{\pi}{4} \right)
\]
(b) From Eq. (8-12)

\[
k = \frac{\omega \sqrt{\mu}}{\varepsilon_0} = \frac{2\pi \times 2 \times 10^9 \times 2}{3 \times 10^8}
\]

\[
= \frac{2\pi}{\lambda}
\]

or

\[
\lambda = \frac{3}{40}m = 7.5cm
\]

8-4 POYNTING VELOCITY AND POWER FLOW

Energy densities in static fields \( \rightarrow \frac{1}{2} \mathbf{D} \cdot \mathbf{E} \) and \( \frac{1}{2} \mathbf{B} \cdot \mathbf{H} \)

Energy is stored in \( \mathbf{E} \) and \( \mathbf{H} \) of an electromagnetic wave.

\[
\uparrow
\]

Same velocity and direction as the wave. \( \uparrow \)

Power delivery

✓ Two curl equations

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{(8-30a)}
\]

\[
\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \text{(8-30b)}
\]

A vector identity

\[
\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}) \quad \text{(8-31)}
\]

Inserting Eq. (8-30) in Eq. (8-31)

\[
-\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{E} \cdot \mathbf{J} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \quad \text{(8-32)}
\]

Assuming \( \varepsilon, \mu \) and \( \sigma \) are constant

\[
\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = \varepsilon \frac{\partial}{\partial t} (\mathbf{E} \cdot \mathbf{E}) = \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{D} \cdot \mathbf{D}) \quad \text{(8-33a)}
\]

\[
\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} = \mu \frac{\partial}{\partial t} (\mathbf{H} \cdot \mathbf{H}) = \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{B} \cdot \mathbf{B}) \quad \text{(8-33b)}
\]

From Eqs. (8-32) and (8-33)

\[
-\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{E} \cdot \mathbf{J} + \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{D} \cdot \mathbf{D}) + \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{B} \cdot \mathbf{B}) \quad \text{: Poynting’s theorem} \quad \text{(8-34)}
\]

Integrating both sides over a volume and applying divergence theorem

\[
-\int_V (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{a} = \int_V \mathbf{E} \cdot \mathbf{J} \, dv + \frac{\partial}{\partial t} \int_V \frac{1}{2} (\mathbf{D} \cdot \mathbf{D}) \, dv + \frac{\partial}{\partial t} \int_V \frac{1}{2} (\mathbf{B} \cdot \mathbf{B}) \, dv \quad \text{(8-35)}
\]

\[
\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow
\]

Ohmic power-loss in \( V \) \n
Increased energy in \( V \) per unit time. \n
Net power flowing into \( V \).

Total instantaneous power flowing into \( V \). \n
(Integrand = power density)
Poynting vector is defined by
\[ \mathbf{S} = \mathbf{E} \times \mathbf{H} \quad : \quad [W / m^2] \] (8-36)

\[ \uparrow \]
\[ \mathbf{S} \perp \mathbf{E} \] and \[ \mathbf{H} \].
\[ \mathbf{S} \parallel \mathbf{k} \]

✔ Time-average power density

\[ \mathbf{S} \sim \mathbf{E}^2 \], a nonlinear function of \[ \mathbf{E} \]
\[ \rightarrow \quad \mathbf{E} \] and \[ \mathbf{H} \] cannot be expressed by phasors in \[ \mathbf{S} \]

Using complex exponential and its complex conjugate.
\[ \mathbf{E} = a_2 \mathbf{E}_o \cos (\omega t - \mathbf{k} \cdot \mathbf{r} + \phi_e) = \frac{1}{2} \left[ a_2 \left( \mathbf{E}_o e^{j\omega t} \right) e^{j\mathbf{k} \cdot \mathbf{r}} + a_2 \left( \mathbf{E}_o e^{-j\omega t} \right) e^{-j\mathbf{k} \cdot \mathbf{r}} \right] \]
\[ = \frac{1}{2} \left[ \left( a_2 \tilde{E}_o \right) e^{-j\mathbf{k} \cdot \mathbf{r}} e^{j\omega t} + \left( a_2 \tilde{E}_o \right) e^{j\mathbf{k} \cdot \mathbf{r}} e^{-j\omega t} \right] \]
\[ = \frac{1}{2} \left[ \mathbf{E}_o e^{-j\mathbf{k} \cdot \mathbf{r}} e^{j\omega t} + \mathbf{E}_o e^{j\mathbf{k} \cdot \mathbf{r}} e^{-j\omega t} \right] \]
\[ = \frac{1}{2} \left[ \mathbf{E} e^{j\omega t} + \mathbf{E}^* e^{-j\omega t} \right] \] (8-37a)

\[ \mathbf{H} = a_n \mathbf{H}_o \cos (\omega t - \mathbf{k} \cdot \mathbf{r} + \phi_h) = \frac{1}{2} \left[ \mathbf{H} e^{j\omega t} + \mathbf{H}^* e^{-j\omega t} \right] \] (8-37b)

where
\[ \mathbf{E}, \] phasor
\[ \mathbf{E}_o, \] vector complex amplitude
\[ \tilde{E}_o, \] scalar complex amplitude
\[ E_o, \] amplitude
\[ \phi_e, \] phase angle
\[ * , \] complex conjugate.

Inserting Eq. (8-37) in Eq. (8-36)
\[ \mathbf{S} = \frac{1}{2} \left[ \mathbf{E} e^{j\omega t} + \mathbf{E}^* e^{-j\omega t} \right] \times \frac{1}{2} \left[ \mathbf{H} e^{j\omega t} + \mathbf{H}^* e^{-j\omega t} \right] \]
\[ = \frac{1}{4} \left[ \mathbf{E} \times \mathbf{H} e^{j2\omega t} + \mathbf{E}^* \times \mathbf{H}^* e^{-j2\omega t} + \mathbf{E} \times \mathbf{H}^* + \mathbf{E}^* \times \mathbf{H} \right] \] (8-38)

Time average of \[ \mathbf{S} \] is defined by
\[ \langle \mathbf{S} \rangle = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \mathbf{S} \, dt \quad : \quad T, \] temporal period of the wave

Time-average Poynting vector
\[ \langle \mathbf{S} \rangle = \frac{1}{2} \text{Re} \left[ \mathbf{E} \times \mathbf{H}^* \right] \] (8-39)
Example 8-3
Given an electromagnetic wave, \( \mathcal{E}(r, t) = a_x E_0 e^{-\beta x} \cos(\omega t - kz) \) and \( \mathcal{H}(r, t) = a_y E_0 \sqrt{\frac{\epsilon}{\mu}} \cos(\omega t - kz) \), find \( \langle S \rangle \) using

(a) \( \mathcal{E} \) and \( \mathcal{H} \)
(b) \( \mathbf{E} \) and \( \mathbf{H} \)

Solution
(a) From Eq. (8-36)
\[
\mathbf{S} = \mathcal{E} \times \mathcal{H} = a_x E_0^2 \sqrt{\frac{\epsilon}{\mu}} \cos^2(\omega t - kz)
\]

Using trigonometry
\[
\mathbf{S} = a_x E_0^2 \sqrt{\frac{\epsilon}{\mu}} \left( 1 + \cos(2\omega t - 2kz) \right)
\]

After time-average
\[
\langle \mathbf{S} \rangle = a_x \frac{E_0^2}{2} \sqrt{\frac{\epsilon}{\mu}} \tag{8-40}
\]

(b) Phasors of \( \mathcal{E} \) and \( \mathcal{H} \)
\[
\mathbf{E} = a_x E_0 e^{-\beta x}
\]
\[
\mathbf{H} = a_y E_0 \sqrt{\frac{\epsilon}{\mu}} e^{-\beta x}
\]

From Eq. (8-39)
\[
\langle \mathbf{S} \rangle = \frac{1}{2} \text{Re} [\mathbf{E} \times \mathbf{H}] = \frac{1}{2} \text{Re} \left( a_x E_0 e^{-\beta x} \times \left( a_y E_0 \sqrt{\frac{\epsilon}{\mu}} e^{-\beta x} \right) \right)
\]
\[
= \frac{E_0^2}{2} \frac{\epsilon}{\mu} \text{Re} \left( (a_x e^{-\beta x}) \times (a_y e^{\beta x}) \right)
\]

Rewriting it
\[
\langle \mathbf{S} \rangle = a_x \frac{E_0^2}{2} \sqrt{\frac{\epsilon}{\mu}} \tag{8-41}
\]

The result is the same as Eq. (8-40).
8-5 POLARIZATION OF PLANE WAVE

Phasor of a uniform plane wave
\[ \mathbf{E} = \mathbf{E}_0 \exp (-j\mathbf{k} \cdot \mathbf{r}) \]
\[ \uparrow \]
\[ \mathbf{E}_0 \perp \mathbf{k} \] from Maxwell
No limits on \( |\mathbf{E}_0| \) and \( \mathbf{a}_{E_0} \)

\( \hat{E}_{ox} \), \( \hat{E}_{oy} \) and \( \hat{E}_{oz} \) of \( \mathbf{E}_0 \)
In phase \[ \rightarrow \] \( \mathbf{E} \) oscillates with time along a straight line
Out of phase \[ \rightarrow \] \( \mathbf{E} \) changes magnitude and direction with time

Orientation of \( \mathbf{E} \) is called the polarization \( \mathbf{H} \) is just given by Eq. (8-23).

8-5.1 Linear Polarization
A plane wave in \(+z\) -direction \[ \rightarrow \] \( \mathbf{E} \) should be in \( xy \) -plane
\[ \mathbf{E} = \mathbf{E}_0 e^{-j\omega t} \left( \hat{E}_{ox} \mathbf{a}_x + \hat{E}_{oy} \mathbf{a}_y \right) e^{-j\omega z} \]
\[ \uparrow \uparrow \]
Scalar complex amplitudes.

\[ \checkmark \] If \( \hat{E}_{ox} = E_{ox} \) and \( \hat{E}_{oy} = E_{oy} \), real values
\[ \mathbf{E} = \text{Re} \left( \left( \mathbf{E}_0 \mathbf{a}_x + E_{oy} \mathbf{a}_y \right) e^{-j\omega z} e^{j\omega t} \right) \]
\[ = \left( E_{ox} \mathbf{a}_x + E_{oy} \mathbf{a}_y \right) \cos (\omega t - \omega z) \]
\[ \uparrow \uparrow \]
\[ \uparrow \]
It oscillates with time along a line parallel to \( E_{ox} \mathbf{a}_x + E_{oy} \mathbf{a}_y \)
\[ \uparrow \]
Linearly polarized in the direction \( E_{ox} \mathbf{a}_x + E_{oy} \mathbf{a}_y \).
Linearly polarized along \( x \) -axis \[ \uparrow \] Linearly polarized along \( y \) -axis \[ \uparrow \]
Two component waves are in phase.

FIGURE 8-3
Example 8-4

Given a linearly polarized wave, \( \mathbf{E} = E_0 \left( \mathbf{a}_x + \sqrt{3} \mathbf{a}_y \right) e^{-j\omega z} \), propagating in free space, find
(a) rotation angle of the polarization vector with respect to \( x \)-axis
(b) time-average power density

Solution

(a) The rotation angle
\[
\theta = \tan^{-1} \left( \frac{\sqrt{3}}{1} \right) = 60^\circ
\]

(b) From Eq. (8-23)
\[
\mathbf{H} = \frac{\varepsilon_0}{\mu_0} \mathbf{a}_x \times E_0 \left( \mathbf{a}_x + \sqrt{3} \mathbf{a}_y \right) e^{-j\omega z} = E_0 \frac{\varepsilon_0}{\mu_0} \left( \mathbf{a}_y - \sqrt{3} \mathbf{a}_x \right) e^{-j\omega z}
\]

From Eq. (8-39)
\[
\langle \mathbf{S} \rangle = \frac{1}{2} \text{Re} \left[ E_x \left( \mathbf{a}_x + \sqrt{3} \mathbf{a}_y \right) e^{-j\omega z} \times \left( E_x \frac{\varepsilon_0}{\mu_0} \left( \mathbf{a}_y - \sqrt{3} \mathbf{a}_x \right) e^{-j\omega z} \right) \right]
\]
\[
= E_0 \frac{\varepsilon_0}{2 \mu_0} \left( \mathbf{a}_x + \sqrt{3} \mathbf{a}_y \right) \times \left( \mathbf{a}_y - \sqrt{3} \mathbf{a}_x \right)
\]
\[
= E_0 \frac{1}{2 \sqrt{\mu_0}} (1 + 3) \mathbf{a}_x
\]

The time-average power density is the sum of those of the waves that are polarized in \( x \) and \( y \)-directions, respectively.

8-5.2 Circular Polarization

\( y \)-component lags behind \( x \)-component by \( 90^\circ \) in time dimension

\[
\mathbf{E} = \left( E_0 \mathbf{a}_x - jE_0 \mathbf{a}_y \right) e^{-j\omega z} = \left( E_0 \mathbf{a}_x + E_0 e^{-j\pi/2} \mathbf{a}_y \right) e^{-j\omega z}
\]

where \( |E_{x0}| = |E_{y0}| = E_0 \).

Instantaneous expression
\[
\mathbf{E} = \mathbf{a}_x E_0 \cos(\omega t - k z) + \mathbf{a}_y E_0 \cos(\omega t - k z - \pi / 2)
\]

At \( z = 0 \)
\[
\mathbf{E} = \mathbf{a}_x E_0 \cos(\omega t) + \mathbf{a}_y E_0 \cos(\omega t - \pi / 2)
\]
\[
= \mathbf{a}_x E_0 \cos(\omega t) + \mathbf{a}_y E_0 \sin(\omega t)
\]

At \( \omega t = 0 \), \( \mathbf{E} = E_0 \mathbf{a}_x \)
At \( \omega t = \pi / 2 \), \( \mathbf{E} = E_0 \mathbf{a}_y \quad \rightarrow \quad \text{Right-hand circularly polarized wave} \]
\[
\uparrow
\]
A circular locus in \( xy \)-plane
Fig. 8-4, right-hand circular polarization
- $y$-comp. lags behind $x$-comp. by $90^\circ$ in time phase
- $y$-comp. arrives at $p$ at a later time than $x$-comp. by a quarter of the temporal period.
- $y$-comp. is displaced backward by a quarter wavelength with respect to the $x$-comp.

The phase relationship is maintained throughout the space at all times.

$y$-comp. leads $x$-comp. by $90^\circ$ in time phase

$\Rightarrow$ Left-hand circularly polarized wave

When the left fingers follow the rotation of the electric field vector, the left thumb points to the propagation direction of the wave.

\[
E = (E_o a_x + jE_o a_y) e^{-j\omega t} = (E_o a_x + E_o e^{j\pi/2} a_y) e^{-j\omega t}
\]  
(8-47)

Instantaneous expression
\[
\mathbf{E} = a_x E_o \cos(\omega t - kz) + a_y E_o \cos(\omega t - kz + \pi / 2)
\]  
(8-48)

At $z = 0$
\[
\mathbf{E} = a_x E_o \cos(\omega t) + a_y E_o \cos(\omega t + \pi / 2)
\]
\[
= a_x E_o \cos(\omega t) - a_y E_o \sin(\omega t)
\]  
(8-49)

\begin{align*}
\text{At } \omega t &= 0, \quad \mathbf{E} = E_o a_x \\
\text{At } \omega t &= \pi / 2, \quad \mathbf{E} = -E_o a_y \quad \Rightarrow \text{Clockwise rotation} \\
&\uparrow \\
&\text{A circular locus in } xy \text{ plane}
\end{align*}
Fig. 8-5, left-hand circular polarization
y - comp. leads x - comp. by 90° in time phase
y - comp. arrives at p at an earlier time than x - comp. by a quarter of the temporal period.
y - comp. is displaced forward by a quarter wavelength with respect to the x - comp.

Example 8-5
A linearly polarized wave can be expressed by the sum of two circularly polarized waves. Find two circularly polarized waves contained in
(a) $E_i = E_o \mathbf{a}_x e^{-j\omega z}$
(b) $E_2 = (E_{ox} \mathbf{a}_x + E_{oy} \mathbf{a}_y) e^{-j\omega z}$

Solution
(a) By adding and subtracting $(1/2) jE_o \mathbf{a}_y$

$$E_i = E_o \mathbf{a}_x e^{-j\omega z} = \frac{1}{2} \left[ (E_o \mathbf{a}_x - jE_o \mathbf{a}_y) + (E_o \mathbf{a}_x + jE_o \mathbf{a}_y) \right] e^{-j\omega z}$$

On the right side, the first term is a right-hand circularly polarized wave and the second is a left-hand circularly polarized wave. Thus, the answer is

$$\frac{1}{2} (E_o \mathbf{a}_x - jE_o \mathbf{a}_y) e^{-j\omega z} \text{ and } \frac{1}{2} (E_o \mathbf{a}_x + jE_o \mathbf{a}_y) e^{-j\omega z}$$

The linearly polarized wave in x - direction is the sum of two circularly polarized waves, with a half the amplitude of the linear polarization, which rotate in the opposite directions and meet each other at x - axis once in every cycle of their rotations.

(b) Magnitude of the amplitude

$$E_o = \sqrt{E_{ox}^2 + E_{oy}^2}$$

Phase angle of the amplitude

$$\theta = \tan^{-1} \frac{E_{oy}}{E_{ox}}$$
The polarization vector is parallel to a “reference line” that is rotated by $\theta$ with respect to $x$-axis. Referring to the result of (a), we can express the linearly polarized wave by the sum of two circularly polarized waves, which rotate in the opposite directions and meet at the reference line once in every cycle of their rotations.

We add a phase $\theta$ to the right-hand circularly polarized wave in Eq. (8-44), and add a phase $-\theta$ to the left-hand circularly polarized wave in Eq. (8-47), so that their polarization vectors become parallel to the reference line at $z = 0$ and $t = 0$,

\[ (E_0 a_x - jE_0 a_y) e^{i\theta} e^{-jka}, \quad (8-51a) \]
\[ (E_0 a_x + jE_0 a_y) e^{-i\theta} e^{-jka}, \quad (8-51b) \]

Reducing the amplitude of the waves in Eq. (8-51) by half and adding two waves

\[ E_2 = \frac{1}{2} (E_0 a_x - jE_0 a_y) e^{i\theta} e^{-jka} + \frac{1}{2} (E_0 a_x + jE_0 a_y) e^{-i\theta} e^{-jka} \]

which is certainly a sum of two circularly polarized waves

Let us check the result in Eq. (8-52). Rearranging it

\[ E_2 = E_0 \left[ \cos \theta a_x + \sin \theta a_y \right] e^{-jka} \]

Since $E_0 \cos \theta = E_{ox}$ and $E_0 \sin \theta = E_{oy}$, it is a linearly polarized wave in the direction that is rotated by $\theta$ with respect to $x$-axis.

\[ 8-5.3 \text{ Elliptical Polarization} \]

Linearly and circularly polarized waves are special cases of an elliptically polarized wave

\[ E = (E_{ox} a_x + E_{oy} e^{i\phi} a_y) e^{-jka} \]

: Elliptically polarized wave

\[ (8-53) \]

Instantaneous expression

\[ \mathbf{s} = E_{ox} a_x \cos (\omega t - kz) + E_{oy} a_y \cos (\omega t - kz + \phi) \]

\[ (8-54) \]
At \( z = 0 \) plane
\[
E_x = E_{ox} \cos(\omega t) \quad (8-55a)
\]
\[
E_y = E_{oy} \cos(\omega t + \varphi) \quad (8-55b)
\]

For \( 0 < \varphi < \pi / 2 \)
At \( \omega t = 0 \), \( E_x = E_{ox} \) and \( E_y = E_{oy} \cos \varphi \) (\( \mathbf{\varepsilon} \) in first quadrant)
At \( \omega t = \pi / 2 \), \( E_x = 0 \) and \( E_y = -E_{oy} \sin \varphi \) (\( \mathbf{\varepsilon} \) parallel to \( -y \)-axis)
\( \uparrow \)
\( \mathbf{\varepsilon} \) is rotating clockwise in \( xy \)-plane

Left-hand elliptically polarized.

\( \checkmark \) Elliptical locus by \( \mathbf{\varepsilon} \) in Eq. (8-54)
New coordinate system, \( x' \) and \( y' \)
\( \uparrow \uparrow \)
Parallel to the major and minor axes

Elliptical locus in general form
\[
E'_{x} = E'_{ox} \cos \left[ - (\omega t + \varphi') \right] = E'_{ox} \cos (\omega t + \varphi') \quad (8-56a)
\]
\[
E'_{y} = E'_{oy} \sin \left[ - (\omega t + \varphi') \right] = -E'_{oy} \sin (\omega t + \varphi') \quad (8-56b)
\]
\( \uparrow \)
ccw rotation in \( x' y' \)-plane

Coord. transformation of Eq. (8-55) to primed coord.
\[
E'_{x} = E_{ox} \cos (\omega t) \cos \theta + E_{oy} \cos (\omega t + \varphi) \sin \theta \quad (8-57a)
\]
\[
E'_{y} = -E_{ox} \cos (\omega t) \sin \theta + E_{oy} \cos (\omega t + \varphi) \cos \theta \quad (8-57b)
\]

By equating Eq. (8-57) with Eq. (8-56)
\[
\tan(2\theta) = \frac{2E_{ox}E_{oy} \cos \varphi}{(E_{ox})^2 - (E_{oy})^2} \quad (8-58)
\]
\[ \rightarrow \]
\[ \tan \theta = E_{oy} / E_{ox} \] for \( \varphi = 0 \), linearly polarized wave.
Example 8-6
Find time-average power density of a uniform plane wave, \( \mathbf{E} = (E_{ox} \mathbf{a}_x + E_{oy} e^{j\theta} \mathbf{a}_y) e^{-j\omega t} \), propagating in free space.

Solution
Vector complex amplitude
\[ \mathbf{E}_a = E_{ox} \mathbf{a}_x + E_{oy} e^{j\theta} \mathbf{a}_y \]

From Eq. (8-23)
\[
\mathbf{H} = \frac{1}{\eta_0} (\mathbf{a}_x \times \mathbf{E}) = \frac{1}{\eta_0} \mathbf{a}_z \times (E_{ox} \mathbf{a}_x + E_{oy} e^{j\theta} \mathbf{a}_y) e^{-j\omega t}
= H_a e^{-j\omega t}
\]

Vector complex amplitude
\[ \mathbf{H}_a = \frac{1}{\eta_0} (E_{ox} \mathbf{a}_y - E_{oy} e^{j\theta} \mathbf{a}_x) \]

Time-average Poynting vector
\[
\langle \mathbf{S} \rangle = \frac{1}{2} \text{Re} \left[ \mathbf{E} \times \mathbf{H}^* \right] = \frac{1}{2} \text{Re} \left[ \mathbf{E}_a \times \mathbf{H}_a^* \right]
= \frac{1}{2\eta_0} \text{Re} \left[ (E_{ox} \mathbf{a}_x + E_{oy} e^{j\theta} \mathbf{a}_y) \times (E_{ox} \mathbf{a}_y - E_{oy} e^{-j\theta} \mathbf{a}_x) \right]
= \frac{1}{2\eta_0} (E_{ox}^2 + E_{oy}^2)
\]

The power delivered by an elliptically-polarized wave is equal to the sum of the individual powers of two linearly-polarized waves.

8-6 PLANE WAVE IN LOSSY MEDIA

✓ An electromagnetic wave in a lossy dielectric.
→ Induced electric dipoles → Power loss
(Same freq. \( \omega \))
↑
Damping force (Collision)

Conduction current also causes power loss.

In a lossy medium, \( \varepsilon = \varepsilon' - j\varepsilon'' \)
↑
Power loss

Magnetic interaction is negligible
→ \( \mu \) is real
8-6.1 Lossy Dielectric (\(\sigma = 0\))

✓ In a simple medium with no free charges and constant \(\varepsilon\) and \(\mu\)

Maxwell’s equations

\[
\begin{align*}
\nabla \times \mathbf{E} &= -j\omega\mu\mathbf{H} \\
\nabla \times \mathbf{H} &= j\omega(\varepsilon' - j\varepsilon)\mathbf{E} = j\omega\varepsilon\mathbf{E} \\
\n\nabla \cdot \mathbf{E} &= 0 \\
\n\nabla \cdot \mathbf{H} &= 0
\end{align*}
\] (8-59a) (8-59b) (8-59c) (8-59d)

Helmholtz’s equation

\( \nabla^2 \mathbf{E} + \mathbf{k}^2 \mathbf{E} = 0 \) (8-60)

Complex wavenumber

\( \mathbf{k} = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu(\varepsilon' - j\varepsilon')} \) (8-61)

Plane wave solution

\[
\mathbf{E} = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}
\] : \( \mathbf{a}_x \) by other means (8-62)

A wave in \( +z \) -direction

\[
\mathbf{E} = \mathbf{E}_0 e^{-j\mathbf{k}_z z} = \mathbf{E}_0 e^{-j\gamma z}
\] (8-63)

Propagation constant

\[
\gamma = j\mathbf{k} = j\omega\sqrt{\mu(\varepsilon' - j\varepsilon')} = \alpha + j\beta
\] (8-64)

where

\( \alpha \), attenuation constant

\( \beta \), phase constant.

\[
\begin{align*}
\alpha &= \omega\sqrt{\frac{\mu\varepsilon'}{2}} \left[ \sqrt{1 + \left(\frac{\varepsilon'}{\varepsilon''}\right)^2} - 1 \right]^{1/2} : \text{Nepers per meter [Np/m]} \\
\beta &= \omega\sqrt{\frac{\mu\varepsilon'}{2}} \left[ \sqrt{1 + \left(\frac{\varepsilon'}{\varepsilon''}\right)^2} + 1 \right]^{1/2} : \text{Radians per meter [rad/m]}
\end{align*}
\] (8-65a) (8-65b)

Loss tangent is defined by

\[
\tan\xi = \frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon}
\] : \( \xi \), loss angle (8-66)

Inserting Eq. (8-64) in Eq. (8-63)

\[
\mathbf{E} = \mathbf{E}_0 e^{-ax} e^{-jbx} = (\mathbf{E}_0 \mathbf{a}_x) e^{-ax} e^{-jbx}
\] (8-67a)
Instantaneous expression
\[ \mathcal{E} = (E_o \mathbf{a}_e) e^{-\alpha z} \cos(\omega t - \beta z) \]  
\[ \uparrow \hspace{1cm} \uparrow \]

Attenuation. Phase variation

✓ Phase difference is \( 2\pi \) between two points separated by \( \lambda \).
\[ \beta \lambda = 2\pi \]

Rewriting it
\[ \beta = \frac{2\pi}{\lambda} \]
: \( \beta \) and \( \lambda \) depend on \( \varepsilon \)

Phase velocity
\[ v_p = \frac{\omega}{\beta} \]  
\[ (8-69) \]

For \( \mu' \varepsilon > \mu \varepsilon_o \)
\( \beta \) in medium \( > k \) in free space
\( \lambda \) in medium \( < \lambda_o \) in free space \( \rightarrow \) \( v_p \) is slower in medium

✓ \( \mathbf{H} \) from Eqs. (8-63) and Eq. (8-59a)
\[ \nabla \times (E_o \mathbf{a}_a e^{-\gamma z}) = -\gamma \mathbf{a}_a \times (E_o \mathbf{a}_a e^{-\gamma z}) \]
\[ = -j\omega \mu \mathbf{H} \]

Rewriting it, with Eq. (8-64)
\[ \mathbf{H} = \frac{\sqrt{\varepsilon}}{\mu} E_o (\mathbf{a}_a \times \mathbf{a}_e) e^{-\gamma z} = \frac{E_o}{\hat{\eta}} (\mathbf{a}_a \times \mathbf{a}_e) e^{-\alpha z} e^{-\beta z} \]  
\[ (8-70) \]

Complex intrinsic impedance, \( \hat{\eta} = \hat{E}_o / \hat{H}_o \)
\[ \frac{\mu}{\varepsilon} \]  
\[ = \frac{\mu}{\varepsilon'} - \frac{j\mu}{\varepsilon'} \]  
\[ (8-71) \]

Complex \( \hat{\eta} \)
\( \rightarrow \) \( \mathcal{E} \) and \( \mathcal{E} \) are out of phase

Example 8-7
From a wave, \( \mathbf{E} = \mathbf{a}_z E_o e^{-0.3z} e^{-j\omega t/2} \), given in a lossy dielectric of \( \mu = \mu_o \) and \( \varepsilon' = 2 \varepsilon_o \), determine intrinsic impedance of the medium.

Solution
Attenuation and phase constants
\( \alpha = 0.3 \)
\( \beta = 0.5 \)

From Eq. (8-65)
\[ \left( \frac{\alpha}{\beta} \right)^2 = \sqrt{1 + \left( \frac{\varepsilon'}{\varepsilon} \right)^2} - 1 \]
\[ = \sqrt{1 + \left( \frac{\varepsilon'}{\varepsilon} \right)^2} + 1 \]  
\[ (8-72) \]
Inserting $\alpha$ and $\beta$ in Eq. (8-72), loss tangent
\[
\frac{\epsilon''}{\epsilon'} = 1.88
\]  

(8-73)

From Eq. (8-71), with $\epsilon' = 2\epsilon_o$ and the loss tangent
\[
\hat{\eta} = \sqrt{\frac{\mu_o}{2\epsilon_o}} \sqrt{1 - j\left(\frac{\epsilon''}{\epsilon'}\right)} = \frac{120\pi}{\sqrt{2}} \frac{1}{\sqrt{1 - j1.88}} = 183e^{0.54}
\]  

(8-74)

$\mathcal{H}$ lags behind $\mathcal{E}$ by 0.54 radian. Note that $\alpha$, $\beta$, and $\hat{\eta}$ are closely related to each other.

8-6.2 Lossy Dielectric ($\sigma \neq 0$)

Conduction current is the dominant source of power loss.

In a dielectric with $\sigma \neq 0$ and no damping
\[
\nabla \times E = -j\omega\mu \mathbf{H}
\]  

(8-75a)

\[
\nabla \times H = \mathbf{J} + j\omega\epsilon \mathbf{E} = (\sigma + j\omega\epsilon) \mathbf{E}
\]  

(8-75b)

\[
\nabla \cdot E = 0
\]  

(8-75c)

\[
\nabla \cdot H = 0
\]  

(8-75d)

$\mathbf{E}$ and $\mathbf{H}$, generated outside $\rightarrow \nabla \cdot \mathbf{D} = \rho_v = 0$, $\mathbf{J}$ is induced

\[
\uparrow
\]

No source charge or current

Rewriting Eq. (8-75b)
\[
\nabla \times H = j\omega\left(\epsilon - j\frac{\sigma}{\omega}\right) \mathbf{E}
\]  

(8-76)

\[
= j\omega(\epsilon' - j\epsilon^*) \mathbf{E} = j\omega\epsilon \mathbf{E}
\]

\[
\epsilon' = \epsilon
\]  

(8-77a)

\[
\epsilon^* = \frac{\sigma}{\omega}
\]  

(8-77b)

Plane wave solution
\[
\mathbf{E} = \mathbf{E}_0 e^{-j\mathbf{k}_0 z}
\]  

(8-78)

Substituting Eq. (8-77) in Eqs. (8-64) and (8-65)
\[
\gamma = \alpha + j\beta = \omega \sqrt{\mu \epsilon} \sqrt{1 - j \frac{\sigma}{\omega\epsilon}}
\]  

(8-79)

\[
\alpha = \omega \sqrt{\frac{\mu \epsilon}{2}} \left[ \sqrt{1 + \left( \frac{\sigma}{\omega\epsilon} \right)^2} - 1 \right]^{1/2}
\]  

(8-80a)

\[
\beta = \omega \sqrt{\frac{\mu \epsilon}{2}} \left[ \sqrt{1 + \left( \frac{\sigma}{\omega\epsilon} \right)^2} + 1 \right]^{1/2}
\]  

(8-80b)
Similarly
\[
\hat{\eta} = \left[ \frac{\mu}{\varepsilon} \right] \left( \frac{1}{1 - j\sigma / \omega\varepsilon} \right)
\]  
(8-81)

✓ Low-loss dielectric, \( \varepsilon'' \ll \varepsilon' \) or \( \sigma / \omega\varepsilon \ll 1 \)

Applying binomial expansion
\[
\gamma \equiv j\sqrt{\frac{\mu}{\varepsilon}} \left[ 1 - j \frac{\sigma}{2\omega\varepsilon} + \frac{1}{8} \left( \frac{\sigma}{\omega\varepsilon} \right)^2 \right]
\]
\[
= \alpha + j\beta
\]  
(8-82)

Binomial expansion in general form
\[
(1 + a)^n = 1 + na + \frac{n(n - 1)}{2!} a^2 + ...
\]
where \(|a| \ll 1\).

From Eq. (8-82)
\[
\alpha \equiv \frac{\sigma}{2} \sqrt{\frac{\mu}{\varepsilon}}
\]  
(8-83a)
\[
\beta \equiv \omega \sqrt{\mu\varepsilon} \left[ 1 + \frac{1}{8} \left( \frac{\sigma}{\omega\varepsilon} \right)^2 \right]
\]  
(8-83b)
where
\[
\alpha \sim \sigma
\]
\[
\beta = \omega \sqrt{\mu\varepsilon} , \text{ perfect dielectric}
\]

Similarly
\[
\hat{\eta} \equiv \left[ \frac{\mu}{\varepsilon} \right] \left[ 1 + j \frac{\sigma}{2\omega\varepsilon} \right]
\]  
(8-84)

Positive phase angle
\( \mathbf{E} \) leads \( \mathbf{H} \) in time phase.

**FIGURE 8-8**
Example 8-8

Given an electromagnetic wave, \( \mathbf{E} = a_x 300e^{-0.21z - j(2.2z)} [V/m] \), propagating in a medium of \( \bar{\eta} = 220 + j21 \ [\Omega] \), find

(a) \( \langle \mathbf{S} \rangle \) at \( z = 0.5m \)
(b) time-average ohmic power-loss per unit volume at \( z = 0.5m \) at steady state

Solution

(a) Magnetic field intensity
\[
\mathbf{H} = \frac{1}{\bar{\eta}} (a_x \times \mathbf{E}) = a_y \frac{300}{220 + j21} e^{-0.21z - j(2.2z)}
\]

Time average Poynting vector
\[
\langle \mathbf{S} \rangle = \frac{1}{2} \text{Re} \left[ \mathbf{E} \times \mathbf{H}^* \right] = a_x \frac{1}{2} \text{Re} \left[ \frac{300^2 e^{-0.42z}}{220 - j21} \right]
\]
\[
= a_x (202.7) e^{-0.42z}
\] (8-85)

At \( z = 0.5m \)
\[
\langle \mathbf{S} \rangle = a_x (202.7) e^{-0.42 \times 0.5} = 164.3 a_x \left[ W/m^2 \right]
\]

(b) Time averaging Poynting’s theorem in Eq. (8-34) at steady state
\[
-\nabla \times \langle \mathbf{E} \times \mathbf{H} \rangle = \langle \mathbf{S} \times \mathbf{J} \rangle
\]
where \( \mathbf{J} \) is the real instantaneous value and \( \mathbf{J} \) is the phasor.

Evaluating the left side of Eq. (8-86) using Eq. (8-85)
\[
-\nabla \cdot \langle \mathbf{S} \rangle = (202.7)(0.42)e^{-0.42z} = 85.1 e^{-0.42z}
\]

At \( z = 0.5m \), time-average ohmic power-loss per unit volume is
\[
-\nabla \cdot \langle \mathbf{S} \rangle = 85.1e^{-0.42 \times 0.5} = 69.0 \left[ W/m^3 \right]
\] (8-87)

In the other method, the power loss can be calculated by the use of \( \mathbf{J} \)
First, from Eqs. (8-79) and (8-81)
\[
\frac{\gamma}{\bar{\eta}} = j \sigma \left( 1 - j \frac{\sigma}{\bar{\omega} \epsilon} \right)
\]

From the equation we obtain
\[
\sigma = \text{Re} \left[ \frac{\gamma}{\bar{\eta}} \right] = \text{Re} \left[ \frac{0.21 + j2.2}{220 + j21} \right] = 1.89 \times 10^{-3} \left[ S/m \right]
\]

At \( z = 0.5m \)
\[
\mathbf{E} = a_x 300e^{-0.21 \times 0.5 - j(2.2 \times 0.5)}
\]
\[
\mathbf{J} = a_x \left( 300 \times 1.89 \times 10^{-3} \right) e^{-0.21 \times 0.5 - j(2.2 \times 0.5)}
\]
Time-average ohmic power-loss
\[
\frac{1}{2} \mathbf{E} \cdot \mathbf{J} = \frac{1}{2} (300^2 \times 1.89 \times 10^{-3}) e^{-0.21+0.5\pi} = 69.0 \left[ W / m^3 \right]
\] (8-88)

Two results in Eqs. (8-87) and (8-88) are identical.

### 8-6.3 Good Conductor

Good conductor \( \rightarrow \sigma / \omega \epsilon \gg 1 \)

\[
\gamma \equiv j \omega \sqrt{\mu \epsilon} \sqrt{-j \frac{\sigma}{\omega \epsilon}} = \frac{1+j}{\sqrt{2}} \sqrt{\omega \mu \epsilon}
\]

\[
\alpha = \beta = \sqrt{\pi f / \mu \sigma}
\] (8-89)

Plane wave in a good conductor
\[
\mathbf{E} = \mathbf{E}_0 e^{-\alpha z} e^{-\beta z}
\]

\[
\mathbf{E}_0 \alpha \beta \pi f \mu \sigma
\] (8-90)

\[
\mathfrak{F} = (\mathbf{E}_0 \alpha \beta \pi f \mu \sigma) e^{-\alpha z} \sqrt{\pi f / \mu \sigma} e^{-\beta z} \cos (\omega t - z \sqrt{\pi f / \mu \sigma})
\] (8-91a)

Attenuation

**Depth of penetration** or **skin depth**

\[
\delta = \frac{1}{\sqrt{\pi f / \mu \sigma}} = \frac{1}{\alpha} = \frac{1}{\beta}
\] : [m] (8-92)

From \( \beta = 2\pi / \lambda \)

\[
\delta = \frac{\lambda}{2\pi}
\] (8-93)

Phase velocity in a good conductor

\[
\nu_p = f \lambda = \omega \delta = 2 \sqrt{\pi f / \mu \sigma}
\]

\[
\nu_p = 1.3 \times 10^4 m / s \text{ at } 1 GHz \text{ in Cu}
\] (8-94)

**FIGURE 8-9**
Example 8-9
Find the skin depth in copper of $\sigma = 5.8 \times 10^7 \text{ [S/m]}$ as a function of frequency

Solution
From Eq. (8-93), with $\mu \approx \mu_0$

$$\delta = \frac{1}{\sqrt{\pi f \mu_0 \sigma}} = \frac{1}{\sqrt{\pi f \times 4\pi \times 10^{-7} \times 5.8 \times 10^7}}$$

The answer is

$$\delta = \frac{0.066}{\sqrt{f}} \text{ [m]}$$

At a frequency of 1GHz, skin depth in copper is only 2.1 $\mu$m.

From Eq. (8-81)

$$\tilde{\eta} \equiv \frac{j\omega \mu}{\sigma}$$

or

$$\tilde{\eta} = \frac{(1 + j)}{\sigma \delta} : \text{E leads H by 45° in time phase} \quad (8-95)$$

The real part of $\tilde{\eta}$

$$R_s = \frac{1}{\sigma \delta} : \text{Surface resistance per unit area} \quad (8-96)$$

Resistance of a good conductor of cross section $w \times \delta$ and length $l$

$$R = \frac{l}{\omega \delta \sigma} \quad (8-97)$$

Example 8-10
A plane wave $\mathbf{E} = (E_0 \mathbf{a}_x) e^{-(1+j)\omega t}$ propagates into a good conduction of conductivity $\sigma$ in the region $z \geq 0$.

(a) Find time-average power loss in a volume $l \times w \times \infty \text{ [m$^3$]}$

(b) Find total current, in phasor form, crossing the area $w \times \infty \text{ [m$^2$]}$ in $yz$-plane

(c) Assuming the current in (b) flows uniformly in a volume of cross section $w \times \delta$ and length $l$, find its resistance, and time-average power loss in the volume

\[ \text{FIGURE 8-10} \]
Solution

(a) Time-average power loss

\[
\langle P \rangle = \int \frac{1}{2} Re[J \cdot E^*] dv = \frac{1}{2} \sigma w t Re \left[ \int_{z=0}^{\infty} E \cdot E^* dz \right] = \frac{1}{2} \sigma w t E_o^2 \int_{z=0}^{\infty} e^{-2z/j} dz = \frac{1}{4} \sigma w t E_o^2
\]

(b) Total current

\[
I = \int_{y=0}^{\infty} \int_{z=0}^{\infty} \sigma E \cdot ds = \int_{z=0}^{\infty} 2 \pi \sigma w E_o e^{-(1+j)z/j} dz = \frac{\sigma w \delta E_o}{1 + j}
\]

(c) The conductor has a cross section \( w \delta \), length \( \ell \) and conductivity \( \sigma \). The resistance

\[
R = \frac{\ell}{w \delta \sigma}
\]

Time-average power loss, with Eqs. (8-99) and (8-100)

\[
\langle P \rangle = R \frac{1}{2} Re[I I^*] = \frac{1}{4} \sigma w \delta E_o^2
\]

The results in Eqs. (8-98) and (8-101) are identical, from which we can assume that the current flows uniformly only in the region \( 0 \leq z \leq \delta \), as far as the total power loss is concerned. We can also obtain the same power loss as Eq. (8-98) or (8-101) from the voltage \( E_o \ell \) that is applied to the resistance \( R \) of Eq. (8-100).

✓ Power density in a conductor

Electric and magnetic field intensities

\[
E = a_x E_o e^{-(1+j)z/j} \delta
\]

\[
H = \frac{1}{\eta} (a_x \times E) = a_y \frac{\sigma \delta}{1 + j} E_o e^{-(1+j)z/j}
\]

Time-average Poynting vector

\[
\langle S \rangle = \frac{1}{2} Re \left[ (a_x E_o e^{-z/j} e^{-jz/j}) \times \left( a_y \frac{\sigma \delta}{1 + j} E_o e^{-z/j} e^{-jz/j} \right) \right] = \frac{1}{2} Re \left[ a_x E_o^2 \frac{\sigma \delta (1 + j)}{2} e^{-2z/j} \right]
\]

or

\[
\langle S \rangle = a_x \frac{\sigma \delta}{4} E_o^2 e^{-2z/j} \text{ : Attenuated}
\]
Example 8-11
Referring to Fig. 8-10, find the total power transmitted into the conductor through an area $w \times l$ of the conductor surface.

Solution
From Eq. (8-102), time-average power density at $z = 0$

\[
\langle S \rangle = a_x \frac{\sigma \delta}{4} E_0^2
\]  

(8-103)

Time-average total power across the area

\[
\langle P \rangle = \int_{z=0}^{z=\infty} \int_0^w \langle S \rangle \cdot ds = \frac{1}{4} \sigma \delta E_0^2 w l
\]  

(8-104)

Note that $ds$ is in $a_x$ direction, which is in the same direction as $\langle S \rangle$, not the outward surface normal. The power across the surface decays to zero at $z = \infty$, as is shown in Eq. (8-102), implying that the transmitted power is dissipated all in the conductor. It is verified by the fact that the results in Eqs. (8-104) and (8-98) are identical.

8-7 PLANE WAVES AT BOUNDARY
A uniform plane wave $\rightarrow$ Reflection
Transmission
↑
Interface
(Different $\varepsilon$ and $\mu$)

8-7.1 Normal Incidence of Plane Wave

FIGUR 8-11
Incident wave polarized in $x$ direction

$$\mathbf{E'} = a_x E_o' e^{-j\beta_1 z} \quad (8-105a)$$

$$\mathbf{H'} = a_y \frac{E_o'}{\eta_1} e^{-j\beta_1 z} \quad (8-105b)$$

where

$\beta_1$, phase constant

$\eta_1$, intrinsic impedance

$E_o'$, amplitude, assumed real

The boundary condition

$\rightarrow (\mathbf{k}_x, \mathbf{k}_z) || \mathbf{a}_z$, uniform $\mathbf{E}_r$ and $\mathbf{E}_t$ at $z = 0$

$\rightarrow (\mathbf{E}_r$ and $\mathbf{E}_t) || \mathbf{a}_z$

The reflected plane wave

$$\mathbf{E'} = a_x E_o' e^{j\beta_1 z} \quad (8-106a)$$

$$\mathbf{H'} = -a_y \frac{E_o'}{\eta_1} e^{j\beta_1 z} \quad (8-106b)$$

↑

Traveling in $-z$ direction.

Same $\beta_1$ and $\eta_1$ as $\mathbf{E}_r$, same medium

The transmitted plane wave

$$\mathbf{E'} = a_x E_o' e^{-j\beta_2 z} \quad (8-107a)$$

$$\mathbf{H'} = a_y \frac{E_o'}{\eta_2} e^{-j\beta_2 z} \quad (8-107b)$$

From the boundary condition at $z = 0$ plane

$$\mathbf{E'}(0) + \mathbf{E'}(0) = \mathbf{E'}(0) \quad (8-108a)$$

$$\mathbf{H'}(0) + \mathbf{H'}(0) = \mathbf{H'}(0) \quad (8-108b)$$

Inserting Eqs. (8-105)-(8-107)

$$E_o' + E_o' = E_o' \quad (8-109a)$$

$$\frac{E_o'}{\eta_1} - \frac{E_o'}{\eta_2} = \frac{E_o'}{\eta_2} \quad (8-109b)$$

By solving the equation

$$E_o' = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} E_o' \quad (8-110a)$$

$$E_o' = \frac{2\eta_2}{\eta_2 + \eta_1} E_o' \quad (8-110b)$$
We define
\[
\Gamma = \frac{E_o'}{E_o} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \quad \text{: Reflection coefficient} \quad (8-111a)
\]
\[
\tau = \frac{E_t'}{E_o} = \frac{2\eta_2}{\eta_2 + \eta_1} \quad \text{: Transmission coefficient} \quad (8-111b)
\]

True only for normal incidence
\[
\Gamma < 0 \text{ for } \eta_2 < \eta_1 \quad \rightarrow \quad a_r = -a_i
\]
\[|\Gamma| \neq 0 \text{ for } \eta_1 \neq \eta_2 \quad \rightarrow \text{ Mandatory reflection}
\]
\[|\Gamma| = 0 \text{ for } \eta_1 = \eta_2 \quad \rightarrow \text{ No reflection}
\]

Eq. (8-111) can be applied even to an interface between two lossy media
\[
\rightarrow \quad \text{Complex } \eta_1 \text{ and } \eta_2
\]
\[
\rightarrow \quad \text{Complex } \Gamma \text{ and } \tau
\]

From Eq. (8-111)
\[
1 + \Gamma = \tau
\]

Valid only for normal incidence

**Example 8-12**

For normal incidence of a uniform plane wave, prove
\[
|\Gamma|^2 + (\eta_1 / \eta_2)|\tau|^2 = 1
\]
from the conservation of energy

**Solution**

Time-average power densities of the incident, reflected and transmitted waves
\[
\langle \mathbf{S}' \rangle = \frac{1}{2} \text{Re} \left[ \mathbf{E}' \times \mathbf{H}'^\ast \right] = a_s \frac{1}{2\eta_1} |E'_s|^2 \quad (8-113a)
\]
\[
\langle \mathbf{S}_r \rangle = -a_s \frac{1}{2\eta_1} |E'_r|^2 = -a_s \frac{1}{2\eta_1} |\Gamma|^2 |E'_s|^2 \quad (8-113b)
\]
\[
\langle \mathbf{S}_t \rangle = a_s \frac{1}{2\eta_2} |E'_t|^2 = a_s \frac{1}{2\eta_2} |\tau|^2 |E'_s|^2 \quad (8-113c)
\]

From the conservation of energy
\[
\left| \langle \mathbf{S}' \rangle \right| = \left| \langle \mathbf{S}_r \rangle \right| + \left| \langle \mathbf{S}_t \rangle \right| \quad (8-114)
\]

Substituting Eq. (8-113)
\[
|\Gamma|^2 + \frac{\eta_1}{\eta_2} |\tau|^2 = 1 \quad (8-115)
\]

Two terms on the left side correspond to the reflected and the transmitted energies relative to the incident energy.
A. Standing Wave Ratio

Total electric field intensity in medium 1
\[
E_i = E' + E'' = a_x E'_x e^{-j\beta_1 z} \left[ 1 + \Gamma e^{j\beta_1 z} \right] \\
= a_x \hat{E}_i
\]  
(8-116)

\(\Gamma\) is complex in general
\[
\Gamma = |\Gamma| e^{j\phi} 
\]  
(8-117)

Rewriting Eq. (8-116) with Eq. (8-117)
\[
\hat{E}_i = E'_x e^{-j\beta_1 z} \left[ 1 + |\Gamma| e^{j(2\beta_1 z + \phi)} \right]
\]  
(8-118)

The maximum amplitude and its location
\[
|\hat{E}_i|_{\text{max}} = E'_x (1 + |\Gamma|) \\
Z_{\text{max}} = -\frac{1}{2\beta_1} (\phi + 2\pi n) 
\]  
(8-119)

(8-120)

The minimum amplitude and its location
\[
|\hat{E}_i|_{\text{min}} = E'_x (1 - |\Gamma|) \\
Z_{\text{min}} = -\frac{1}{2\beta_1} (\phi + 2\pi n + \pi) 
\]  
(8-121)

(8-122)

The intermaximal distance is \(2\pi / 2\beta_1\), a half wavelength of the wave in medium 1.

Rewriting Eq. (8-118)
\[
\hat{E}_i = E'_x e^{-j\beta_1 z} \left[ 1 - |\Gamma| + |\Gamma| e^{j(2\beta_1 z + \phi)} \right] \\
= E'_x e^{-j\beta_1 z} \left[ 1 - |\Gamma| + |\Gamma| e^{j\phi} \right] \\
\]  
(8-123)

Real instantaneous expression
\[
\hat{E}_i = E'_x \left( 1 - |\Gamma| \right) \cos (\omega t - \beta_1 z) + 2E'_x |\Gamma| \cos (\beta_1 z + \phi / 2) \cos (\omega t + \phi / 2) 
\]  
(8-124)

\[\uparrow\]  
Traveling wave  
\[\uparrow\]  
Standing wave
Traveling wave, a blue colored line; standing wave, black colored line
\( \omega t = 0, \pi / 4 \) and \( \pi \) for \(|\Gamma| = 0.5\) and \( \phi = 0\)

\[ E \]

The standing wave ratio is defined by
\[ S = \frac{E_{\text{max}}}{E_{\text{min}}} = \frac{1 + |\Gamma|}{1 - |\Gamma|} \]

: Dimensionless

\(|\Gamma|\) ranges from 0 to 1, and \( S \) ranges from 1 to \( \infty \).
Example 8-13
A normally incident wave is reflected off an interface at $z = 0$ plane. Measured in medium 1 are the standing wave ratio of 4, a distance of 0.2m between the first maximum of the electric field intensity and the interface, and a distance of 0.5m between two adjacent maxima. Find the intrinsic impedance $\eta_2$ of medium 2.

Solution
The intermaximal distance is equal to a half wavelength of the wave in medium 1
\[
\lambda = 2 \times 0.5m
\]
or
\[
\beta_1 = \frac{2\pi}{\lambda} = 2\pi.
\]

The first maximum corresponds to $n = 0$ in Eq. (8-120)
\[
z_{\text{max}} = -\frac{1}{2\beta_1} \phi = -0.2m
\]

Combining two equations
\[
\phi = 0.8\pi \quad (8-126)
\]

From Eq. (8-125)
\[
|\Gamma| = \frac{S - 1}{S + 1} = \frac{4 - 1}{4 + 1} = 0.6
\]

The reflection coefficient, from Eqs. (8-126) and (8-127)
\[
\Gamma = 0.6e^{j0.8\pi} \quad (8-128)
\]

Rewriting Eq. (8-111a)
\[
\frac{\eta_2}{\eta_1} = \frac{1 + \Gamma}{1 - \Gamma} \quad (8-129)
\]

Inserting Eq. (8-128) and $\eta_1 = \eta_o$ in Eq. (8-129)
\[
\frac{\eta_2}{\eta_o} = \frac{1 + 0.6e^{j0.8\pi}}{1 - 0.6e^{j0.8\pi}} = \frac{0.99e^{j0.60}}{1.22e^{-j0.23}} = 0.81e^{j0.83}
\]

The answer is
\[
\eta_2 = 305e^{j0.83} [\Omega]
\]

B. Interface Involving Perfect Conductor

Interface between a lossless medium and a perfect conductor
\[
\uparrow \quad z < 0 \quad \uparrow \quad z \geq 0
\]

Perfect conductor $\rightarrow \quad \sigma = \infty \quad \rightarrow \quad \eta = 0$
\[
\Gamma = -1 \quad \text{and} \quad \tau = 0
\]
Incident plane wave in medium 1
\[ \mathbf{E}' = a_x E_o e^{-j \beta_o z} \] : Polarized in \( x \)-direction (8-130a)
\[ \mathbf{H}' = a_y \frac{E_o'}{\eta_i} e^{-j \beta_o z} \] (8-130b)

Reflected plane wave propagates in \(-z\) direction
\[ \mathbf{E}' = a_x \Gamma E_o e^{j \beta_o z} = -a_x E_o e^{j \beta_o z} \] (8-131a)
\[ \mathbf{H}' = a_y \frac{E_o'}{\eta_i} e^{-j \beta_o z} \] (8-131b)

where
Amplitude of \( E' \) from \( E_o' = \Gamma E_o \)
\( \mathbf{H}' \) from Eq. (8-23).

FIGURE 8-14
\[ r = 0 \text{ at conductor surface. Nodes with a period of } \lambda / 2. \]
First node of \( \mathbf{H} \) at \( \lambda / 4 \) from the conductor surface. The same period as \( \mathbf{H} \)
Total field in medium 1
\[ \mathbf{E}_1 = \mathbf{E}' + \mathbf{E}' = a_x E'_x [e^{-j\beta z} - e^{j\beta z}] = -a_x j2E'_x \sin(\beta z) \] (8-132a)
\[ \mathbf{H}_1 = \mathbf{H}' + \mathbf{H}' = a_y E'_y \eta_1 \left[ e^{-j0z} + e^{j0z} \right] = a_y \frac{2E'_y}{\eta_1} \cos(\beta z) \] (8-132b)

Instantaneous expression
\[ \mathbf{E}' = a_x 2E'_x \sin(\beta z) \cos(\omega t - \pi / 2) = a_x 2E'_x \sin(\beta z) \sin(\omega t) \] : Standing wave (8-133a)
\[ \mathbf{H}' = a_y \frac{2E'_y}{\eta_1} \cos(\beta z) \cos(\omega t) \] (8-133b)

Standing wave cannot transport energy in space, because the factor \( -j \) in Eq. (8-132a) makes \( \mathbf{E} \times \mathbf{H}' \) be imaginary and results in \( \mathbf{S} = 0 \).

Example 8-14
A right-hand circularly polarized wave, \( \mathbf{E}' = (E'_x a_x - jE'_y a_y)e^{-j\beta z} \), propagates in free space and impinges on the surface of a perfect conductor at \( z = 0 \) plane. Find
(a) polarization vector of the reflected wave
(b) surface current density on the conductor surface
(c) \( \mathbf{E} \) in free space

Solution
(a) Intrinsic impedance of the perfect conductor
\( \eta_2 = 0 \)

Reflection coefficient
\( \Gamma = -1 \)

Reflected wave
\[ \mathbf{E}' = \Gamma \left( E_o a_x - jE_o a_y \right) e^{j\beta z} = \left( -E_o a_x + jE_o a_y \right) e^{j\beta z} \] (8-134)

Real instantaneous expression for the reflected wave
\[ \mathbf{E}' = \text{Re} \left[ \left( -E_o a_x + e^{j0z} E_o a_y \right) e^{j\beta z} e^{j\omega t} \right] \]
\[ = -a_x E_o \cos(\omega t + \beta z) + a_y E_o \cos \left( \omega t + \beta z + \frac{\pi}{2} \right) \]
\[ = -a_x E_o \cos(\omega t + \beta z) - a_y E_o \sin(\omega t + \beta z) \]

Let us check \( \mathbf{E}' \) at \( z = 0 \) plane
At \( \omega t = 0 \), \( \mathbf{E}' = -a_x E_o \)
At \( \omega t = \pi / 2 \), \( \mathbf{E}' = -a_y E_o \) (8-135)

When the left fingers follow \( \mathbf{E}' \) in Eq. (8-135), the left thumb points to the propagation direction of \( \mathbf{E}' \), or the \(-z\) direction. Since the \( x \) and \( y \) components have the same amplitude, the reflected wave is left-hand circularly polarized.
If a right-hand circularly polarized wave is reflected, it becomes a left-hand circularly polarized wave, or vice versa.

![Diagram](image)

**FIGURE 8-15**

(b) Magnetic field intensities of the incident and reflected waves

\[
\mathbf{H'} = \frac{1}{\eta} (\mathbf{a}_x \times \mathbf{E'}) = \frac{1}{\eta_0} \mathbf{a}_x \times (E_y \mathbf{a}_x - jE_z \mathbf{a}_y) e^{-j\beta z} \\
= \frac{E_y}{\eta_0} (\mathbf{a}_y + j \mathbf{a}_x) e^{-j\beta z}
\]

\[
\mathbf{H'} = \frac{1}{\eta} (\mathbf{a}_x \times \mathbf{E'}) = \frac{1}{\eta_0} (\mathbf{a}_x) \times (-E_x \mathbf{a}_x + jE_y \mathbf{a}_y) e^{j\beta z} \\
= \frac{E_x}{\eta_0} (\mathbf{a}_y + j \mathbf{a}_x) e^{j\beta z}
\]

Total field at \( z = 0 \) plane
\[
\mathbf{H} = \mathbf{H'} + \mathbf{H'} = \frac{2E_x}{\eta_0} (\mathbf{a}_y + j \mathbf{a}_x)
\]

Surface current density
\[
\mathbf{J}_s = \mathbf{a}_n \times \mathbf{H} = (-\mathbf{a}_x) \times \frac{2E_x}{\eta_0} (\mathbf{a}_y + j \mathbf{a}_x) \\
= \frac{2E_x}{\eta_0} (\mathbf{a}_x - j \mathbf{a}_y)
\]

where \( \mathbf{a}_n \) is the unit surface normal.

(c) Total field in free space
\[
\mathbf{E} = \mathbf{E'} + \mathbf{E'} = (E_x \mathbf{a}_x - jE_y \mathbf{a}_y) e^{-j\beta z} + (-E_x \mathbf{a}_x + jE_y \mathbf{a}_y) e^{j\beta z} \\
= -\mathbf{a}_x 2E_x \sin(\beta z) - \mathbf{a}_y 2E_y \sin(\beta z)
\]

Instantaneous expression
\[
\mathfrak{g} = \text{Re} \left[ \mathbf{E} e^{j\omega t} \right] = \text{Re} \left[ \mathbf{a}_x 2E_x \sin(\beta z) e^{j(\omega t - \pi/2)} - \mathbf{a}_y 2E_y \sin(\beta z) e^{j\omega t} \right] \\
= 2E_x \sin(\beta z) [\mathbf{a}_x \sin(\omega t) - \mathbf{a}_y \cos(\omega t)]
\]