Chapter 1. Vector Analysis

1.1 Scalars and Vectors

Scalar: A quantity represented by a single real number
Distance, time, temperature, voltage, etc

Vector: Magnitude and direction
Force, velocity, flux, etc

At a given position and time
a scalar field (function) → A magnitude
(Temperature distribution in a room)
a vector field (function) → A magnitude and a direction
(Movement of smoke particles)

Function conversion between different coordinate systems
Physical laws should be independent of the coordinate systems.
Coordinate system is chosen by convenience

Three main topics
(1) Vector algebra: addition, subtraction, multiplication
(2) Orthogonal coordinate system: Cartesian, Cylindrical, Spherical
(3) Vector calculus: differentiation, integration (gradient, divergence, curl)

1.2 Vector Algebra

A vector has a magnitude and a direction
\[ \vec{A} = A\hat{a}_A \]

Magnitude, \[ |\vec{A}| \]

Unit vector, \[ \frac{\vec{A}}{|\vec{A}|} \]

Graphical representation

Two vectors are equal if they have the same magnitude and direction, even though they may be displaced in space.
• Vector addition, \( \vec{C} = \vec{A} + \vec{B} \)
Two vectors, \( \vec{A} \) and \( \vec{B} \), form a plane

Parallelogram rule : \( \vec{C} \) is the diagonal of the parallelogram
Head-To-Tail rule : The head of \( \vec{A} \) touches the tail of \( \vec{B} \).
\( \vec{C} \) is drawn from the tail of \( \vec{A} \) to the head of \( \vec{B} \).

![Parallelogram rule](image1.png)
![Head-To-Tail rule](image2.png)

Note \( \vec{C} = \vec{A} + \vec{B} = \vec{B} + \vec{A} \) \hspace{1cm} Commutative law
\( \vec{A} + (\vec{B} + \vec{F}) = (\vec{A} + \vec{B}) + \vec{F} \) \hspace{1cm} Associative law

The location of a vector is at the tail of the arrow.

• Vector subtraction
\( \vec{A} - \vec{B} = \vec{A} + (-\vec{B}) \), \hspace{1cm} where \( -\vec{B} = (-\hat{a}_b)|\vec{B}| \)

![Vector subtraction](image3.png)

• Vector multiplication by a scalar
Multiplication of a vector by a positive scalar
\( k\vec{A} = (k\vec{A})\hat{a}_A \) \hspace{1cm} : Change in the magnitude but not the direction

Associative and distributive laws apply
\[ (r + s)(\vec{A} + \vec{B}) = r(\vec{A} + \vec{B}) + s(\vec{A} + \vec{B}) = r\vec{A} + r\vec{B} + s\vec{A} + s\vec{B} \]

• Vector division by a scalar
\( \vec{A} \rightarrow \) Multiplication by the reciprocal of that scalar

• Two vectors are equal, \( \vec{A} = \vec{B} \), if \( \vec{A} - \vec{B} = 0 \)

• Addition and subtraction of vector fields should be done locally.
\( \vec{C}(\vec{r}) = \vec{A}(\vec{r}) + \vec{B}(\vec{r}) \) \hspace{1cm} : Addition at the same position \( \vec{r} \).

Example; magnetic fields of the earth and a horseshoe magnet.
1.3 The Rectangular (Cartesian) Coordinate System

Right-hand rule for $x$, $y$, $z$ directions, Fig (a).
A point is represented by $x$, $y$, $z$ coordinates, Fig (b).

Distance from the origin along $x$-axis

A point can be also represented by a common intersection of three planes, $x=x_1$, $y=y_1$, and $z=z_1$.

Three differential lengths, $dx$, $dy$ and $dz$, from $P$ to $P'$

Three coordinates of $P'$ are $x+dx$, $y+dy$ and $z+dz$ in Fig(c).

The distance between $P$ and $P' = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$

$dx$, $dy$ and $dz$ form a rectangular parallelepiped with a differential volume and six differential areas.

\[ dV = dx\, dy\, dz \]
1.4 Vector Components and Unit Vectors

A point in space is represented by a **position vector** \( \vec{r} = \vec{x} + \vec{y} + \vec{z} \).

\[ \begin{align*}
\vec{x} &= \hat{x} \hat{a}_x, \\
\vec{y} &= \hat{y} \hat{a}_y, \\
\vec{z} &= \hat{z} \hat{a}_z
\end{align*} \]

Three component vectors along each corresponding axis.

Component vectors are given by unit vectors as \( \vec{x} = |\vec{x}| \hat{a}_x, \vec{y} = |\vec{y}| \hat{a}_y \) and \( \vec{z} = |\vec{z}| \hat{a}_z \).

The position vector \( \vec{r}_p \) is from the origin to the point \( P(1,2,3) \).
\[ \vec{r}_p = \hat{a}_x + 2\hat{a}_y + 3\hat{a}_z \]

**Example** A vector from \( P \) to \( Q \)
\[ \vec{R}_{PQ} = \vec{r}_Q - \vec{r}_P = (2\hat{a}_x - 2\hat{a}_y + \hat{a}_z) - (\hat{a}_x + 2\hat{a}_y + 3\hat{a}_z) = \hat{a}_x - 4\hat{a}_y - 2\hat{a}_z \]

- Any vector \( \vec{B} = B_x \hat{a}_x + B_y \hat{a}_y + B_z \hat{a}_z \)
- The magnitude of \( \vec{B} \) : \( |\vec{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2} \)
- The unit vector of \( \vec{B} \) : \( \hat{a}_B = \frac{\vec{B}}{|\vec{B}|} = \frac{\vec{B}}{\sqrt{B_x^2 + B_y^2 + B_z^2}} \)
1.5 The Vector Field

A vector field \( \mathbf{v}(r) \) is a vector function of a position vector \( r \).

If we move from the origin to a point in space following the position vector \( r \), the vector field gives a vector at that point.

Using component vectors

\[
\mathbf{v}(r) = v_x(r) \mathbf{\hat{a}}_x + v_y(r) \mathbf{\hat{a}}_y + v_z(r) \mathbf{\hat{a}}_z
\]

1.6 The Dot Product

The dot product or scalar product of two vectors \( \mathbf{A} \) and \( \mathbf{B} \) are defined as

\[
\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \cos \theta_{AB}
\]

where \( \theta_{AB} \) is the smaller angle between \( \mathbf{A} \) and \( \mathbf{B} \).

Note

\[
\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}
\]

The dot product obeys the distributive law

\[
\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z
\]

Using \( a_i a_j = 0 \) for \( i \neq j \), \( a_i a_j = 1 \) for \( i = j \)

Note

\[
\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2 = A^2
\]
Projection of a vector $\vec{B}$ in $\hat{a}$ direction

It is defined as

$$\vec{B} \cdot \hat{a} = |\vec{B}| |\hat{a}| \cos \theta_{Ba} = |\vec{B}| \cos \theta_{Ba}$$

---

**Example 1.2**

In order to illustrate these definitions and operations, let us consider the vector field $\vec{G} = y\hat{a}_x - 2.5x\hat{a}_y + 3\hat{a}_z$ and the point $Q(4, 5, 2)$. We wish to find $\vec{G}$ at $Q$: the scalar component of $\vec{G}$ at $Q$ in the direction of $\hat{a}_N = \frac{1}{\sqrt{3}}(2\hat{a}_x + \hat{a}_y - 2\hat{a}_z)$; the vector component of $\vec{G}$ at $Q$ in the direction of $\hat{a}_N$; and finally, the angle $\theta_{Ga}$ between $\vec{G}(\vec{r}_Q)$ and $\hat{a}_N$.

**Solution.** Substituting the coordinates of point $Q$ into the expression for $\vec{G}$, we have

$$\vec{G}(\vec{r}_Q) = 5\hat{a}_x - 10\hat{a}_y + 3\hat{a}_z$$

Next we find the scalar component. Using the dot product, we have

$$\vec{G} \cdot \hat{a}_N = (5\hat{a}_x - 10\hat{a}_y + 3\hat{a}_z) \cdot \frac{1}{\sqrt{3}}(2\hat{a}_x + \hat{a}_y - 2\hat{a}_z) = \frac{1}{\sqrt{3}}(10 - 10 - 6) = -2$$

The vector component is obtained by multiplying the scalar component by the unit vector in the direction of $\hat{a}_N$,

$$(\vec{G} \cdot \hat{a}_N)\hat{a}_N = -(2)\frac{1}{\sqrt{3}}(2\hat{a}_x + \hat{a}_y - 2\hat{a}_z) = -1.333\hat{a}_x - 0.667\hat{a}_y + 1.333\hat{a}_z$$

The angle between $\vec{G}(\vec{r}_Q)$ and $\hat{a}_N$ is found from

$$\vec{G} \cdot \hat{a}_N = |\vec{G}| \cos \theta_{Ga}$$

$$-2 = \sqrt{25 + 100 + 9} \cos \theta_{Ga}$$

and

$$\theta_{Ga} = \cos^{-1}\left(\frac{-2}{\sqrt{134}}\right) = 99.9^\circ$$

**Example** Law of Cosine

Use vectors to prove

$$\vec{C} = \vec{B} - \vec{A}$$

$$C^2 = \vec{C} \cdot \vec{C} = (\vec{B} - \vec{A}) \cdot (\vec{B} - \vec{A})$$

$$\Rightarrow \vec{B} \cdot \vec{B} + \vec{A} \cdot \vec{A} - \vec{A} \cdot \vec{B} - \vec{B} \cdot \vec{A}$$

$$\Rightarrow A^2 + B^2 - 2AB \cos \alpha$$
1.7 The Cross Product

The cross product or vector product of two vectors $\vec{A}$ and $\vec{B}$ are defined as

$$\vec{A} \times \vec{B} = \hat{a}_n |\vec{A}| |\vec{B}| \sin \theta_{AB}$$

$\hat{a}_n$ is a unit vector normal to the surface formed by $\vec{A}$ and $\vec{B}$

Note

$$|\vec{A} \times \vec{B}|$$ represents the area of the parallelepiped.

$$\vec{B} \times \vec{A} = -(\vec{A} \times \vec{B})$$

$$\hat{a}_x \times \hat{a}_y = \hat{a}_z, \quad \hat{a}_y \times \hat{a}_z = \hat{a}_x, \quad \hat{a}_z \times \hat{a}_x = \hat{a}_y$$

(1)

- Evaluation of $\vec{A} \times \vec{B}$

$$\vec{A} \times \vec{B} =$$

$$A_x B_y (\hat{a}_x \times \hat{a}_y) + A_y B_x (\hat{a}_y \times \hat{a}_x) + A_z B_x (\hat{a}_z \times \hat{a}_x) +$$

$$A_x B_y (\hat{a}_y \times \hat{a}_x) + A_y B_z (\hat{a}_y \times \hat{a}_z) + A_y B_x (\hat{a}_z \times \hat{a}_x) +$$

$$A_z B_x (\hat{a}_z \times \hat{a}_x) + A_z B_y (\hat{a}_z \times \hat{a}_y) + A_z B_x (\hat{a}_z \times \hat{a}_z)$$

Using (1)

$$\vec{A} \times \vec{B} = \hat{a}_x (A_y B_z - A_z B_y) + \hat{a}_y (A_z B_x - A_x B_z) + \hat{a}_z (A_x B_y - A_y B_x)$$

or

$$\vec{A} \times \vec{B} =$$

$$\begin{bmatrix}
\hat{a}_x & \hat{a}_y & \hat{a}_z \\
A_x & A_y & A_z \\
B_x & B_y & B_z
\end{bmatrix}$$
1.8 Cylindrical Coordinate System

A point is given by the intersection of three mutually perpendicular surfaces.

- A circular cylinder, \( \rho = \text{constant} \)
- A half plane, \( \phi = \text{constant} \)
- A plane, \( z = \text{constant} \)

Unit vectors are defined along the direction of increasing coordinate values, Fig (b). Its direction is perpendicular to the surface of constant coordinate value.

\( \hat{a}_\rho \) at \( P(\rho, \phi, z) \) is directed radially outward and normal to the cylindrical surface \( \rho = \rho_1 \).

\( \hat{a}_\phi \) at \( P(\rho, \phi, z) \) is normal to the plane \( \phi = \phi_1 \).

\( \hat{a}_z \) at \( P(\rho, \phi, z) \) is normal to the plane \( z = z_1 \).

Note

\( \hat{a}_\rho \) and \( \hat{a}_\phi \) are functions of position, NOT CONSTANTS.

Three unit vectors are mutually perpendicular with cyclic order

\[ \hat{a}_\rho \times \hat{a}_\phi = \hat{a}_z \]

- Differential volume

Increase \( \rho \rightarrow \rho + d\rho, \phi \rightarrow \phi + d\phi, z \rightarrow z + dz \)

The six surfaces enclose a differential volume, Fig(c). It can be approximated as a rectangular parallelepiped. The lengths of the sides are \( d\rho, \rho d\phi \) and \( dz \).

\[ d\phi \] itself is an angle not a length

The volume is given by \( \rho d\rho d\phi dz \)

Areas of side surfaces are given by \( \rho d\rho d\phi, \rho d\phi dz \) and \( d\rho dz \).
• Coordinate transformation of a point
\[ x = \rho \cos \phi \quad \rho = \sqrt{x^2 + y^2} \]
\[ y = \rho \sin \phi \quad \phi = \tan^{-1}(y/x) \]
\[ z = z \quad \zeta = \zeta \]

• Coordinate transformation of a vector
\[ \vec{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z \Rightarrow \vec{A} = A_\rho \hat{a}_\rho + A_\phi \hat{a}_\phi + A_z \hat{a}_z \]
\[ \hat{a}_x \cdot \vec{A} \hat{a}_x \cdot \vec{A} \hat{a}_x \cdot \vec{A} \equiv \]
\[ \hat{a}_y \cdot \vec{A} \hat{a}_y \cdot \vec{A} \hat{a}_y \cdot \vec{A} \equiv \]
\[ \hat{a}_z \cdot \vec{A} \hat{a}_z \cdot \vec{A} \hat{a}_z \cdot \vec{A} \equiv \]

<table>
<thead>
<tr>
<th>( \hat{a}_x \cdot \vec{A} )</th>
<th>( \hat{a}_y \cdot \vec{A} )</th>
<th>( \hat{a}_z \cdot \vec{A} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \cos \phi )</td>
<td>( -\sin \phi )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \sin \phi )</td>
<td>( \cos \phi )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 1 )</td>
</tr>
</tbody>
</table>

**Example 1.3**

Transform the vector \( \mathbf{B} = y \mathbf{a}_x - x \mathbf{a}_y + z \mathbf{a}_z \) into cylindrical coordinates.

**Solution.** The new components are
\[ B_\rho = \mathbf{B} \cdot \hat{a}_\rho = y(\mathbf{a}_x \cdot \hat{a}_\rho) - x(\mathbf{a}_y \cdot \hat{a}_\rho) \]
\[ = y \cos \phi - x \sin \phi = \rho \sin \phi \cos \phi - \rho \cos \phi \sin \phi = 0 \]
\[ B_\phi = \mathbf{B} \cdot \hat{a}_\phi = y(\mathbf{a}_x \cdot \hat{a}_\phi) - x(\mathbf{a}_y \cdot \hat{a}_\phi) \]
\[ = -y \sin \phi - x \cos \phi = -\rho \sin^2 \phi - \rho \cos^2 \phi = -\rho \]

Thus,
\[ \mathbf{B} = -\rho \hat{a}_\phi + z \hat{a}_z \]
1.9 Spherical Coordinate System

A point is given by the intersection of three mutually perpendicular surfaces.

A sphere, \( r = \text{constant} \)
A cone, \( \theta = \text{constant} \)
A half plane, \( \phi = \text{constant} \)

Unit vectors are normal to these surfaces and directed toward increasing \( r, \theta \) and \( \phi \).

They are mutually perpendicular and have a cyclic order
\[
\hat{a}_r \times \hat{a}_\theta = \hat{a}_\phi
\]

- Differential volume
  Increase \( r \to r + dr, \theta \to \theta + d\theta, \phi \to \phi + d\phi \)

The distance between two spheres is \( dr \)
- two cones is \( rd\theta \)
- two half planes is \( r \sin \theta d\phi \)

The differential volume is \( r^2 \sin \theta \, dr \, d\theta \, d\phi \).

The differential areas are \( r \, dr \, d\theta, r \sin \theta \, dr \, d\phi \) and \( r^2 \sin \theta \, d\theta \, d\phi \).
• Coordinate transformation of a point

\[
\begin{align*}
    x &= r \sin \theta \cos \phi \\
    y &= r \sin \theta \sin \phi \\
    z &= r \cos \theta
\end{align*}
\]

\[
\begin{align*}
    r &= \sqrt{x^2 + y^2 + z^2} \\
    \theta &= \cos^{-1} \left( \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right) \\
    \phi &= \tan^{-1} \left( \frac{y}{x} \right)
\end{align*}
\]

• Coordinate transformation of a vector

It can be done by using the projection of a vector.

<table>
<thead>
<tr>
<th>Table 1.2</th>
<th>Dot products of unit vectors in spherical and rectangular coordinate systems</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_r )</td>
<td>( \sin \theta \cos \phi )</td>
</tr>
<tr>
<td>( a_\theta )</td>
<td>( \sin \theta \sin \phi )</td>
</tr>
<tr>
<td>( a_\phi )</td>
<td>( \cos \theta )</td>
</tr>
</tbody>
</table>

**Example 1.4**

We illustrate this transformation procedure by transforming the vector field \( \mathbf{G} = (xz/y) \mathbf{a}_r \) into spherical components and variables.

**Solution.** We find the three spherical components by dotting \( \mathbf{G} \) with the appropriate unit vectors, and we change variables during the procedure:

\[
G_r = \mathbf{G} \cdot \mathbf{a}_r = \frac{xz}{y} \mathbf{a}_r \cdot \mathbf{a}_r = \frac{xz}{y} \sin \theta \cos \phi \\
= r \sin \theta \cos \theta \frac{\cos^2 \phi}{\sin \phi}
\]

\[
G_\theta = \mathbf{G} \cdot \mathbf{a}_\theta = \frac{xz}{y} \mathbf{a}_\theta \cdot \mathbf{a}_\theta = \frac{xz}{y} \cos \theta \cos \phi \\
= r \cos^2 \theta \frac{\cos^2 \phi}{\sin \phi}
\]

\[
G_\phi = \mathbf{G} \cdot \mathbf{a}_\phi = \frac{xz}{y} \mathbf{a}_\phi \cdot \mathbf{a}_\phi = \frac{xz}{y} (-\sin \phi) \\
= -r \cos \theta \cos \phi
\]

Collecting these results, we have

\[
\mathbf{G} = r \cos \theta \cos \phi (\sin \theta \cot \phi \mathbf{a}_r + \cos \theta \cot \phi \mathbf{a}_\theta - \mathbf{a}_\phi)
\]

Appendix A describes the general curvilinear coordinate system of which the rectangular, circular cylindrical, and spherical coordinate systems are special cases. The first section of this appendix could well be scanned now.